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Debt-Ridden Borrowers and Economic Slowdown

Keiichiro Kobayashi and Daichi Shirai [*]

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Abstract

Economic growth slows for an extended period after a financial crisis. We construct a model in which a one-time buildup of debt can depress the economy persistently, even when there is no financial technology shock. A noteworthy policy implication of this model is that relief from excessive debt can restore economic growth. This contrasts with the findings of existing models, which do not admit debt reduction as an effective cure for persistent stagnation. We consider the debt dynamics of firms under endogenous borrowing constraints, with lenders having an option to forgive defaulting borrowers. A firm is referred to as *debt-ridden* when it owes maximum debt, and pays all income in each period as an interest payment. In the deterministic case, a debt-ridden firm continues inefficient production permanently. Further, if the initial debt exceeds a certain threshold, the firm intentionally chooses to increase borrowing and, thus, to become debt-ridden. The emergence of a substantial number of debt-ridden firms lowers economic growth persistently by reducing the growth rate of aggregate productivity. In this case, because lenders have no incentive to reduce excessive debt, government interventions may be necessary.

JEL Classification Numbers: E30, G01, O40

Keywords: Endogenous borrowing constraint, debt overhang, secular stagnation, labor wedge

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1 Introduction

The decade after a financial crisis tends to be associated with low economic growth (Reinhart and Rogoff, 2009; Reinhart and Reinhart, 2010). In this period, growth in total factor productivity (TFP) also slows, and can even become negative (Kehoe and Prescott, 2007). A growing concern related to the economic slowdown after a financial crisis is that of “secular stagnation.” This occurred in the aftermath of the Great Recession of 2007–2012, causing concern that the US and European economies may stagnate persistently in the coming decades (Summers, 2013; Eggertsson and Mehrotra, 2014). It has also been pointed out that financial constraints were tightened, both during and after the Great Recession (e.g., Altavilla, Darracq Paries and Nicoletti, 2015). However, which factors caused the tightening of these financial constraints, and whether this can cause a persistent slowdown in economic growth remain unclear. In this study, we propose a theoretical model in which the buildup of debt tightens borrowing constraints and lowers growth persistently.

In existing models, persistent recessions are usually caused by persistent shocks. See, for example, Christiano, Eichenbaum and Trabandt (2015) for the Great Recession, Cole and Ohanian (2004) for the Great Depression, and Kaihatsu and Kurozumi (2014) for the lost decade of Japan. In our study, the buildup of debt, which is a one-time shock, can change the economy persistently. Many authors have argued that persistent shocks that cause persistent recessions are exogenous changes in the parameters of financial technology, such as the risk shock in Christiano, Motto and Rostagno (2014) and the financial shock in Jermann and Quadrini (2012). In this study, we consider that an exogenous shock increases firms’ debt substantially, whereas there is no change in the structural parameters of financial technology. The increase in the amount of debt tightens borrowing constraints persistently, even though there is no technological change. Thus, the shock we consider can be understood as a redistribution shock, redistributing wealth from borrowers to lenders as a one-time shock. One example of such a redistribution shock is the boom and bust of the asset-price bubble, which changed the value of collateral for debt.

Our model is a version of those with endogenous borrowing constraints, similar to Jermann and Quadrini (2012), in which lenders can choose whether to liquidate defaulting borrowers or to forgive them. There is a distinction between inter-period and intra-period loans in this economy. The borrowing constraint binds more tightly as the initial amount of inter-period debt increases. As the borrowing constraint tightens, firms cannot raise sufficient intra-period debt for working capital, which leads to inefficient production. When the initial debt reaches the maximum sustainable amount, firms fall into a “debt-ridden” state, in which they can repay only the interest, even if they pay all income in each period. As a result, the amount of debt does not decrease. Even though debt-ridden firms have a maximum amount of debt, they can continue production, because the borrowing constraint does not bind too tightly. It does not bind too tightly because the lender can unilaterally seize a portion of the output when the borrower defaults, before they start renegotiating. Production becomes permanently inefficient for debt-ridden firms in the deterministic case. Moreover, when the initial debt exceeds a certain threshold, the firm chooses to increase borrowing and intentionally becomes debt-ridden. This result implies that an overly indebted firm may rationally choose to become, and then stay, debt-ridden forever.

We embed the model of firms' debt into a general equilibrium model of endogenous growth, where productivity grows as a result of firms' R&D activities. If a substantial number of firms become debt-ridden, aggregate borrowing capacity declines persistently. Tighter borrowing constraints discourage firms' R&D activities and make productivity growth persistently low. They also diminish the labor wedge persistently. These features of our model seem to be consistent with the facts observed in persistent recessions after financial crises (see Section 2).

Our contribution to the literature is to show that the buildup of debt can persistently tighten borrowing constraints and, thus, can cause aggregate inefficiency, even if there is no technological shock. Thus, our model implies a policy recommendation distinct from those of most existing models, in which exogenous shocks on technological parameters cause persistent recessions and the policymaker can only mitigate the shocks by conducting accommodative monetary and fiscal policies or by designing ex-ante financial regulations. In our model, debt restructuring or debt forgiveness for overly indebted borrowers restores aggregate efficiency and enhances economic growth. Note that restoring economic growth does not necessitate the physical liquidation of debt-ridden firms, but rather their relief from excessive debt.

Intuitive example: To illustrate how persistent inefficiency can arise in our model, we consider a simple model of a firm that produces output $f(\sigma)$ from input σ, where $f'(\sigma) > 0$ and $f''(\sigma) < 0$. The first best that maximizes the social surplus, $f(\sigma) - \sigma$, is attained by σ^*, where σ^* solves $f'(\sigma) = 1$. This firm initially holds debt b_{-1}, and then borrows new debt $\frac{b}{R}$, where R is the loan rate. Thus, the firm's cash flow is $\pi = f(\sigma) - \sigma - b_{-1} + \frac{b}{R}$. Here, we assume that b_{-1} and b are given exogenously, such that $\pi \geq 0$. The firm chooses σ to maximize $f(\sigma) - \sigma$, subject to the borrowing constraint

$$\sigma \leq \phi f(\sigma) + \max\left\{\xi S - \frac{b}{R},\ 0\right\},$$

where ϕ and ξS are constants, satisfying $0 \leq \phi < 1$ and $\xi S > 0$. This borrowing constraint is derived in Section 3. Considering the Lagrangean $L(\sigma, \mu) = f(\sigma) - \sigma +$

$\mu\left[\phi f(\sigma) - \sigma + \max\{\xi S - \frac{b}{R},\ 0\}\right]$, the first-order condition (FOC) is equal to

$$f'(\sigma) = \frac{1+\mu}{1+\phi\mu}.$$

When the Lagrange multiplier with respect to the borrowing constraint is positive ($\mu > 0$), the above FOC implies $f'(\sigma) > 1$, which means that input is smaller than σ^* and production becomes inefficient. To make this example interesting, we assume that the constant ξS satisfies $\sigma^* < \phi f(\sigma^*) + \xi S$. If $\frac{b}{R}$ is smaller than ξS, the borrowing constraint becomes

$$\sigma \leq \phi f(\sigma) + \xi S - \frac{b}{R}. \tag{1}$$

This borrowing constraint is qualitatively similar to those in Jermann and Quadrini (2012) and Kiyotaki and Moore (1997). If b is small, such that $0 \leq b \leq b^*$, where $b^* = [\phi f(\sigma^*) - \sigma^* + \xi S]R$, then the first best is achieved: $\sigma = \sigma^*$. If $b^* < b < R\xi S$, the borrowing constraint binds tightly ($\mu > 0$) and production becomes inefficient. However, in the dynamic model, the firm repays as much of its debt as possible in order to reduce b and to relax the borrowing constraint. Thus, when the borrowing constraint is (1), inefficiency is temporary. This result is similar to those of existing models of borrowing constraints. When debt $\frac{b}{R}$ is larger than ξS, the borrowing constraint becomes

$$\sigma \leq \phi f(\sigma). \tag{2}$$

We define σ_z as the solution to $\sigma = \phi f(\sigma)$. Assuming that ϕ is sufficiently small, we posit that $\sigma_z < \sigma^*$ and production is inefficient. The Lagrange multiplier μ_z is calculated from $f'(\sigma_z) = \frac{1+\mu_z}{1+\phi\mu_z}$ (> 1). Thus, we can show that inefficiency can continue permanently if b_{-1} is sufficiently large, as follows. We define $b_z = \frac{f(\sigma_z)-\sigma_z}{1-\frac{1}{R}}$ and assume that $\frac{b_z}{R} > \xi S$. When the borrowing constraint is (2) (i.e., $\frac{b}{R} > \xi S$), the cash flow is $\pi = f(\sigma_z) - \sigma_z - b_{-1} + \frac{b}{R}$. The value of b must satisfy $\pi \geq 0$. Suppose that $b_{-1} = b_z$. In this case, b must be chosen such that $b \geq b_z$ in order to satisfy $\pi \geq 0$. Therefore, in the dynamic model, once the debt to be repaid in the current period (b_{-1}) is greater than or equal to b_z, the debt to be repaid in the next period (b) must also be greater than or equal to b_z, and this chain continues forever. Then, the borrowing constraint continues to be (2), and production is permanently inefficient ($\sigma = \sigma_z$). In summary, when $b_{-1} < b_z$, inefficiency continues temporarily, but continues permanently if $b_{-1} \geq b_z$. In this model, permanent inefficiency arises from the accumulation of debt, not from changes in financial or production technology.

Related literature: Our theory is related to the literature on debt overhang, such as Myers (1977), Krugman (1988), and Lamont (1995). The debt overhang problem typically causes inefficiency in the short run. In this study, inefficiency can continue permanently. Our study is also closely related to the work of Caballero, Hoshi and Kashyap (2008).

They analyze "zombie lending," defined as the provision by banks of a de facto subsidy to unproductive firms, and argue that congestion by zombie firms hinders the entry of more productive firms and lowers aggregate productivity. In this study, we make a complementary point to their argument: even an intrinsically productive firm can become unproductive when it is debt-ridden. This point results in a notably different policy implication: Caballero, Hoshi and Kashyap (2008) implied that the physical liquidation of zombie firms is desirable to restore a higher level of productivity. However, our theory implies that zombie firms can restore a high level of productivity if they are relieved of excessive debt. In the macroeconomic literature, endogenous borrowing constraints were introduced by the seminal work of Kiyotaki and Moore (1997).[1] Then, endogenous borrowing constraints in an economy where intra-period and inter-period loans exist were analyzed by Albuquerque and Hopenhayn (2004), Cooley, Marimon and Quadrini (2004), and Jermann and Quadrini (2006, 2007, 2012). The modeling method in this study is closest to that of Jermann and Quadrini (2012), especially in derivation of the borrowing constraint, which we formulate as the no-default condition, as they do. However, an important difference between the two studies is that we assume that the lender can unilaterally seize a portion of the output when the borrower defaults, before they start renegotiating. Our model is also closely related to that of Kobayashi and Nakajima (2015), who analyzed endogenous borrowing constraints and nonperforming loans (NPLs). Furthermore, our model is similar to that of Guerrón-Quintana and Jinnai (2014) in that a temporary shock persistently affects productivity growth, although there is a significant difference in the policy implications. In our model, the emergence of debt-ridden borrowers due to a redistribution shock causes a persistent recession. Thus, debt restructuring (i.e., wealth redistribution from lenders to borrowers) restores aggregate efficiency. In contrast, debt restructuring has no effect in the model of Guerron-Quintana and Jinnai because, in their model, the financial crisis is caused by a shock to the parameters of financial technology. Another study closely related to ours is Ikeda and Kurozumi (2014). They built a medium-scale DSGE model with financial friction, à la Jermann and Quadrini (2012), and endogenous productivity growth, à la Comin and Gertler (2006). Their study is different from ours in that Ikeda and Kurozumi (2014) also posited a financial crisis as an exogenous technological shock.

The remainder of this paper is organized as follows. In the next section, we review the facts on persistent recessions after financial crises. Section 3 presents the partial equilibrium model of the lender–borrower relationship and analyzes the debt dynamics. In Section 4, we construct the full model by embedding the financial frictions of the previous section into an endogenous growth model, showing that stagnation can continue persis-

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[1] Kobayashi and Shirai (2016) analyzed the effects of wealth redistribution on an economy using a similar borrowing constraint model.

tently when a substantial number of debt-ridden borrowers emerge. Section 5 presents our concluding remarks.

2 Facts on persistent recessions after financial crises

There exist numerous examples of decade-long stagnation after a financial crisis. The most notable episode was the Great Depression in the 1930s in the United States, along with the similar depressions during that period globally. Ohanian (2001) highlighted the large productivity decline during the US Great Depression that is unexplained by usual demand-side factors, such as capital utilization and labor hoarding. Kehoe and Prescott (2007) drew our attention to the fact that many countries have experienced decade-long recessions, which they called the "great depressions" of the 20th century. The studies presented in their book unanimously emphasized that declines in the growth rate of TFP were the primary cause of these great depressions.

A well-known example of a decade-long recession was the 1990s in Japan, where the growth rates of GDP and TFP in the 1990s were both lower than those in the 1980s. The kink is apparent at the beginning of the 1990s, when huge asset-price bubbles burst in the stock and real estate markets. Table 1 presents various estimates of the TFP growth rate in Japan. Hayashi and Prescott (2002) emphasized that growth in TFP slowed in the 1990s. Kobayashi and Inaba (2006) confirmed their result using the business cycle accounting method. Fukao and Miyagawa (2008) estimated TFP using an industry-level data set called the Japan Industrial Productivity (JIP) database, and confirmed the substantial TFP slowdown in the 1990s.

One notable feature in the 1990s in Japan was the significant decrease in entries and the increase in exits of firms. Figure 1 compares the entry and exit of firms in Japan and in the United States. In the literature, the procyclicality of net entry is well known (Bilbiie, Ghironi and Melitz, 2012). Net entry also contributes significantly to TFP growth for US manufacturing establishments (Bartelsman and Doms, 2000).[2]

Another characteristic of Japan in the 1990s that may be related to the productivity slowdown was persistently lingering non-performing loans (NPLs) in the banking sector. NPLs represent the excess debt of nonfinancial firms, mainly in the real estate, wholesale, retail, and construction sectors. Figure 2 shows the NPLs in Japan from 1992 to 2012. The delayed disposal of huge NPLs was seen as a de facto subsidy to non-viable firms (i.e., zombie lending). This zombie lending has also been considered to be the cause of Japan's persistent recession (Peek and Rosengren, 2005; Caballero, Hoshi and Kashyap, 2008).

A labor wedge reduction is also a characteristic of an economic slowdown in the af-

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[2] Nishimura, Nakajima and Kiyota (2005) argued that the malfunctioning entries and exits contributed substantially to the fall in Japan's TFP in the late 1990s.

	HP	KI	JIP2014
1971–1980	0.83		2.04
1981–1990	1.93	2.87	2.02
1991–2000	0.36	0.58	0.03
2001–2005		0.90	1.39
2006–2011			-0.28

Table 1: TFP growth rate in Japan

Note: HP, KI, and JIP2014 are taken from updated versions of Hayashi and Prescott (2002), Kobayashi and Inaba (2006) (revised data), and Fukao and Miyagawa (2008), respectively.

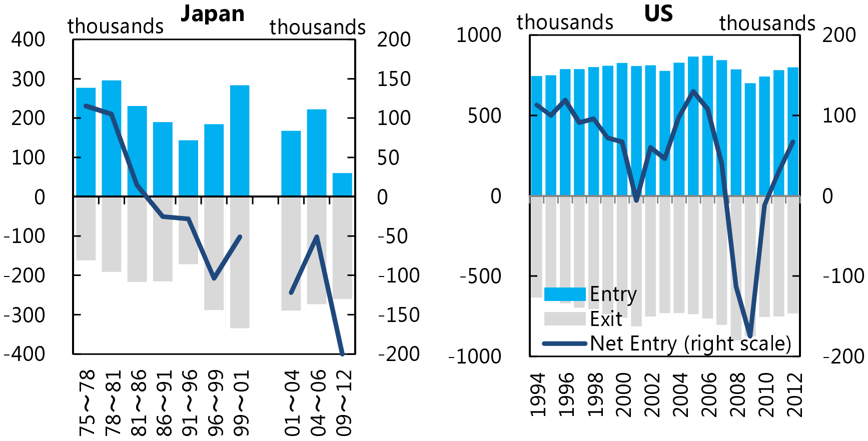

Figure 1: Entry and exit of private sector establishments: United States and Japan

Note: Japan's figures after 2001 are based on the 1993-basis industry classification.

Sources: (Japan) Ministry of Internal Affairs and Communications, "Establishment and Enterprise Census;" (US) Bureau of Labor Statistics, "Business Employment Dynamics."

termath of a financial crisis. Recently, a growing body of literature on business cycle accounting (Chari, Kehoe and McGrattan, 2007) has analyzed various episodes of business fluctuations, including decade-long stagnations. Business cycle accounting focuses on four wedges as the driving forces of business cycles: the efficiency wedge, labor wedge, investment wedge, and government wedge.[3] Chari, Kehoe and McGrattan (2007) noted that reductions in the efficiency wedge and labor wedge were the two primary factors that drove the Great Depression of the 1930s. Kobayashi and Inaba (2006) and Otsu (2011) emphasized the same factors for the lost decade of Japan in the 1990s. The macroeconomic literature has recently focused considerable attention on the effects of a reduction in the labor wedge in recessions (see Mulligan, 2002; Shimer, 2009). A sharp decline in

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[3] The efficiency wedge is the observed TFP; the labor wedge represents market frictions that manifest as an imaginary labor income tax in a prototype real business cycle model; the investment wedge represents market frictions that manifest as an imaginary investment tax; and the government wedge is a deadweight loss, which manifests as government consumption.

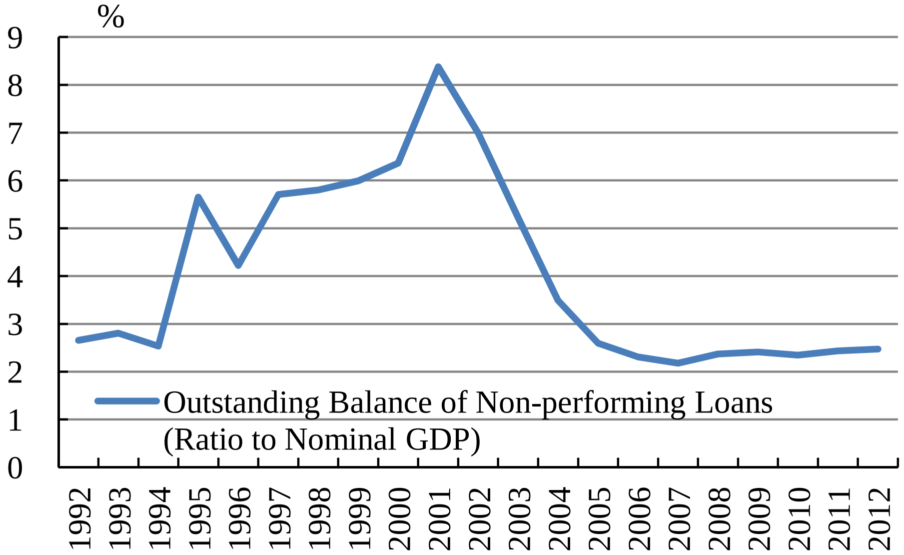

Figure 2: Development of NPLs

Note: These NPLs are the risk management loans defined in the Banking Act in Japan. They consist of loans to bankrupt borrowers, delayed loans, three-month overdue loans, and loans with modified terms and conditions. Risk management loans do not include securitized loans.
Sources: Financial Services Agency,*Status of Non-Performing Loans*; Cabinet Office, Government of Japan,*Annual Report on National Accounts.*

the labor wedge was also observed in the US economy during the final crisis of 2007–2009 (Kobayashi, 2011; Pescatori and Tasci, 2011).

3 A model of corporate debt with borrowing constraints

In this section, we consider the partial equilibrium model of debt contracts. We derive the borrowing constraint and analyze the debt dynamics under exogenously given prices. We then embed this model into the endogenous growth model in Section 4.

3.1 Setup

Time is discrete and continues from 0 to infinity: $t = 0, 1, 2, \cdots, \infty$. There are three agents in this model: a bank (lender), a firm (borrower), and a household (worker). The main players are the bank and the firm; the household only supplies labor and capital at the market prices and buys consumer goods from the firm. Real prices $\{w_t,\ r_t^K,\ r_t,\ m_t\}$, for $t = 0, 1, 2, \cdots, \infty$, are taken as given, where w_t is the wage rate, r_t^K is the rental rate of capital, r_t is the inter-temporal rate of interest for safe assets, and m_t is the stochastic discount factor. The stochastic discount factor is later determined in the general equilibrium model, and satisfies

$$\frac{1}{1+r_t} = \mathbb{E}_t \frac{m_{t+1}}{m_t},$$

where $\mathbb{E}_t$ is the expectation operator at t, and m_t is defined as $m_t = \beta^t \frac{\partial}{\partial C_t} U(C_t, L_t)$, where β is the subjective time discount factor for the household, C_t is consumption, L_t is the labor supply, $U(C, L)$ is the period utility, and $\frac{\partial}{\partial C} U(C, L)$ is the differentiation of utility with respect to C (see Section 4).

Consumer goods are produced by the firm from the labor and capital inputs. The firm's gross revenue in period t is given by

$$F(A_t, k_t, l_t) = A_t k_t^{\alpha\eta} l_t^{(1-\alpha)\eta},$$

where k_t and l_t are the capital and labor inputs, respectively, chosen in period t, A_t is a time-variant revenue parameter, and $0 < \eta < 1$. Firms use equity and debt, where debt is not state-contingent. We focus on the case with initial debt stock b_{-1} at $t = 0$, where $\frac{b_{-1}}{R_{-1}}$ is the amount of inter-period debt at the end of the previous period, and R_t is the gross rate of corporate loans.[4] In this study, following Jermann and Quadrini (2012), we assume that firms hold inter-period debt because it offers tax advantages.[5] Thus, R_t is determined by

$$R_t = 1 + (1 - \tau) r_t,$$

where τ represents the tax benefit. Debt $\frac{b_{t-1}}{R_{t-1}}$ at the end of period $t - 1$ grows at gross rate R_{t-1} to become b_{t-1} at the beginning of period t.

In period t, the firm employs labor l_t and capital k_t from the household to produce and sell consumer goods, and earns revenue $F(A_t, k_t, l_t) = A_t k_t^{\alpha\eta} l_t^{(1-\alpha)\eta}$. The cost of the capital and labor inputs for the firm is

$$\sigma_t = r_t^K k_t + w_t l_t.$$

The firm needs to borrow working capital, σ_t, from the bank as an intra-period loan and pay the household in advance of production.[6] When production is completed, the firm

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[4] Why does the firm hold debt initially? In this model, the initial debt is given as an exogenous shock. The following is one explanation, which falls outside our model, for why the initial debt is generated. The debt in this model can be interpreted as unsecured debt. Suppose that a firm puts up a valuable asset (e.g., real estate or corporate stocks) as collateral and the corporate debt b_{-1} is initially completely secured by this collateral. Then, the asset-price bubble collapses in period 0 and the value of the collateral asset decreases to, say, zero, meaning that b_{-1} becomes unsecured debt. In this way, the bubble collapse can generate (unsecured) debt b_{-1}.

[5] This assumption is a short cut to formulate the motivation to have debt. As is well known, with asymmetric information and costly state verification, the optimal contract does take the form of debt (e.g., Townsend, 1979; Gale and Hellwig, 1985).

[6] The reason why the firm does not use inter-period debt to finance working capital for production is the limited commitment due to agency problems. Suppose the firm pays the wage for production of period $t+1$ in period t: in this case, the worker cannot commit to provide the labor input in period $t+1$. Suppose the firm saves a part of inter-period borrowing $\frac{b_t}{R_t}$ in the form of safe assets, with the intention to use it in period $t + 1$ for working capital. In this case, employees in the firm can easily steal and consume the safe asset privately in period t and the firm cannot use it for working capital in period $t + 1$.

receives revenue $F(A_t, k_t, l_t)$. Let us denote the set of exogenous state variables at t by x_t. Here, $x_t = \{A_t, r_t^K, w_t, r_t, m_t\}$. We denote the space of x_t by Λ (i.e., $x_t \in \Lambda$). State x_t follows a Markov process with the transition function $Q(x_t, dx_{t+1})$. Thus, the expectation of a variable $q(x_t, x_{t+1})$ is given by

$$\mathbb{E}_t[q(x_t, x_{t+1})] \equiv \int q(x_t, x_{t+1}) Q(x_t, dx_{t+1}).$$

We define $f(\sigma_t, x_t)$ by

$$f(\sigma_t, x_t) = \max_{k,l} F(A_t, k, l),$$
$$\text{subject to } \; r_t^K k + w_t l \leq \sigma_t.$$

Thus, the solution implies

$$f(\sigma_t, x_t) = A_t \left(\frac{\alpha}{r_t^K}\right)^{\alpha\eta} \left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\eta} \sigma_t^{\eta}.$$

The budget constraint for the firm is given by

$$\pi_t \leq f(\sigma_t, x_t) - \sigma_t - b_{t-1} + \frac{b_t}{R_t}, \tag{3}$$

where π_t is the payment to the firm owner as a dividend. The payment of intra-period loan $\sigma_t = w_t l_t + r_t^K k_t$ is subject to the following borrowing constraint (derived in the next subsection):

$$\sigma_t \leq \phi f(\sigma_t, x_t) + \max \left\{\xi S(x_t) - \frac{b_t}{R_t},\ 0\right\}, \tag{4}$$

where $0 \leq \phi < 1,\ \ 0 \leq \xi \leq 1$, and $S(x_t)$ is defined by (7), which is the value that the lender can obtain by foreclosure of the firm. Throughout this paper, we assume that

$$\phi < \eta,$$

which means production becomes inefficient when the borrowing constraint is $\sigma_t \leq \phi f(\sigma_t)$. The firm-owner is an entrepreneur who has no liquid assets in hand and is protected by limited liability, as in Albuquerque and Hopenhayn (2004). Therefore, the dividend must be non-negative:

$$\pi_t \geq 0. \tag{5}$$

The firm cannot avoid the limited liability constraint (5) by soliciting equity investment from outside investors because it takes time to sell corporate shares in the market, whereas the funds are needed immediately.[7]

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[7] The limited liability constraint can be interpreted as a variant of the re-saleability constraint on corporate shares, as in Kiyotaki and Moore (2012). We obtain similar results if we use the re-saleability constraint instead of the limited liability constraint.

Now, we can describe the optimization problem for the firm. Denoting the value of the firm with debt stock b_{t-1} at the beginning of period t as $V(b_{t-1}, x_t)$, the firm's problem is written as the following Bellman equation:

$$V(b_{t-1}, x_t) = \max_{b_t, \sigma_t, \pi_t} \pi_t + \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} V(b_t, x_{t+1}) \right], \tag{6}$$

subject to the budget constraint (3), borrowing constraint (4), limited liability constraint (5), and participation constraint of the bank (i.e., the No-Ponzi condition)

$$b_t \leq b_z(x_t),$$

where $\frac{b_z(x_t)}{R_t}$ is the upper limit of the amount that the bank agrees to lend in period t, given by (8) below.

Equilibrium condition: In the above problem (6), the firm takes $\{b_z(x_t), S(x_t)\}_{t=1}^{\infty}$ as given. In equilibrium, $S(x)$ is determined endogenously from the solution to the firm's problem. The value of foreclosure $S(x_t)$ is the maximum value that the bank can obtain from operating the seized firm by itself. Thus, in equilibrium, the following must be satisfied:

$$S(x_t) = \max_{b} \ \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} V(b, x_{t+1}) \right] + \frac{b}{R_t}. \tag{7}$$

Upper limit of debt: The upper limit of debt $b_{z,t} = b_z(x_t)$ is the upper bound of repayable debt under any future realizations of the states, $\{x_{t+j}\}_{j=1}^{\infty}$, given that x_t is the current state.[8] We derive $b_z(x_t)$ on the premise that $\frac{b_z(x_t)}{R(x_t)} > \xi S(x_t)$, which is later justified by the parameter restriction (9). Under this assumption, the borrowing constraint becomes $r_t^K k_t + w_t l_t \leq \phi F(A_t, k_t, l_t)$, when the outstanding debt is $b_z(x_t)$. We define $\{k_{z,t}, l_{z,t}\}$ by

$$\begin{aligned} \{k_{z,t}, l_{z,t}\} &= \arg\max F(A_t, k_t, l_t) \\ \text{subject to } \ & r_t^K k_t + w_t l_t = \phi F(A_t, k_t, l_t). \end{aligned}$$

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[8] When the amount of debt is larger than $\frac{b_{z,t}}{R_t}$, there is a positive probability of default. Thus, the bank agrees to lend only if the amount of debt is no greater than $\frac{b_{z,t}}{R_t}$, given that the loan rate is equal to the market rate for safe assets, $1 + r_t$. In the general case where the loan rate can be set larger than $1 + r_t$, the bank may agree to lend a larger amount than $\frac{b_{z,t}}{R_t}$, as long as the expected rate of return is no smaller than $1 + r_t$. In this case, the upper limit of debt can be larger than $\frac{b_{z,t}}{R_t}$. However, even with a larger upper limit, our analyses and results presented in this paper do not change qualitatively.

Define $\mu_{z,t}$ as the Lagrange multiplier associated with the above constraint. Since $F(A, k, l) = Ak^{\alpha\eta} l^{(1-\alpha)\eta}$, these variables are given by

$$\begin{aligned}
k_{z,t} &= \left[\phi A_t \left(\frac{1-\alpha}{w_t} \right)^{(1-\alpha)\eta} \left(\frac{\alpha}{r_t^K} \right)^{1-(1-\alpha)\eta} \right]^{\frac{1}{1-\eta}}, \\
l_{z,t} &= \left[\phi A_t \left(\frac{1-\alpha}{w_t} \right)^{1-\alpha\eta} \left(\frac{\alpha}{r_t^K} \right)^{\alpha\eta} \right]^{\frac{1}{1-\eta}}, \\
\mu_{z,t} &= \frac{\eta - \phi}{(1-\eta)\phi}.
\end{aligned}$$

We define $\sigma_{z,t} = \sigma_z(x_t)$ as the solution to $\sigma = \phi f(\sigma, x_t)$. Then, it is easily confirmed that

$$\begin{aligned}
\sigma_{z,t} &= r_t^K k_{z,t} + w_t l_{z,t}, \\
f(\sigma_{z,t}, x_t) &= F(A_t, k_{z,t}, l_{z,t}).
\end{aligned}$$

Define, recursively, that for a given value of x_{t-1},

$$b_z(x_{t-1}) = \inf_{x_t \in \Lambda(x_{t-1})} (1-\phi) f(\sigma_z(x_t), x_t) + \frac{b_z(x_t)}{R_t}, \tag{8}$$

where $\Lambda(x_{t-1})$ is the domain for x_t, given that the state of time $t-1$ is x_{t-1}. If $\Lambda(x_{t-1}) = \Lambda$, then the upper limit is a constant (i.e., $b_z(x_{t-1}) = b_z$). If the condition

$$\xi S(x_t) < \frac{b_z(x_t)}{R_t}, \quad \text{for all} \ \ x_t, t$$

is satisfied, then it is justified that $b_z(x_t)$ is truly the maximum repayable debt. To ensure the above condition, we assume the following restriction (9) on the parameters.

Definition 1. The first-best amount of input, $\sigma^*(x)$, is defined as the solution to

$$\frac{\partial}{\partial \sigma} f(\sigma, x) = 1.$$

The upper limit of the total surplus of the match between the bank and firm, given that the current state is x, is denoted by $\omega(x)$, for $x \in \Lambda$, and the unconditional upper limit is denoted by $\bar{\omega}$. These are defined as follows:

$$\begin{aligned}
\omega(x_t) &= [1 + \tau r(x_t)] \{ f(\sigma^*(x_t), x_t) - \sigma^*(x_t) \} + \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \omega(x_{t+1}) \right], \\
\bar{\omega} &\equiv \sup_{x \in \Lambda} \omega(x).
\end{aligned}$$

Since the maximum amount of tax advantage is $\tau r(x) \{ f(\sigma^*(x), x) - \sigma^*(x) \}$, it is obvious that $S(x) \leq \omega(x)$, because $S(x)$ is the value to the bank of operating the seized firm. Thus, we assume the following restriction on the parameter values:

$$\xi \omega(x_t) < \frac{b_z(x_t)}{R_t}, \quad \text{for all} \ \ x_t. \tag{9}$$

This inequality ensures that the maximum repayable debt is strictly larger than the value that the bank can obtain by foreclosure. This is because the bank can recover at most $\xi\omega(x_t)$ by foreclosure, where $\xi\,(<1)$ is the probability that the firm can be operated successfully without the human capital of the original firm-owner (see also Section 3.2). Thus, we assume that the bank never offers debt-for-equity conversion to the borrower, even if it increases the total surplus. This assumption is justified by (9) if we further assume that the firm-owner has full bargaining power.

Technological difference between inter- and intra-period debt: We have the following difference between inter-period debt b_{t-1} and intra-period debt σ_t. The firm has the chance to default on inter-period debt b_{t-1} at the beginning of period t, and it will do so if and only if the continuation value of the firm is negative, $V(b_{t-1}, x_t) < 0$. However, this never happens because, when $b_{t-1} \leq b_z(x_{t-1})$, the firm's dividend is non-negative ($\pi(x_t) \geq 0$), as is the continuation value ($V(b_{t-1}, x_t) \geq 0$). Thus, as long as $b_{t-1} \leq b_z(x_{t-1})$, the firm never defaults on inter-period debt b_{t-1}. The firm has the chance to default on intra-period debt σ_t at the end of period t, which we analyze in the next subsection, where the borrowing constraint (4) for σ_t is given as the no-default condition. Thus, the firm does not default on σ_t in equilibrium.

Timing of events: The events in a given period t occur in the following way. The firm and bank enter period t with debt outstanding b_{t-1}.[9] At the beginning of the period, the firm has the chance to default on b_{t-1}, and it will do so if the continuation value is negative (which never happens, because $b_{t-1} \leq b_z(x_{t-1})$). Then, the firm borrows intra-period debt σ_t, employs labor and capital by paying σ_t, and produces output $f(\sigma_t, x_t)$. The firm repays b_{t-1} and borrows new inter-period debt $\frac{b_t}{R_t}$ by paying $b_{t-1} - \frac{b_t}{R_t}$. Finally, it repays the intra-period debt σ_t to the bank. At this point, the firm has the chance to default on σ_t. After repaying σ_t, the firm pays out the remaining amount, $\pi_t = f(\sigma_t, x_t) - \sigma_t - b_{t-1} + \frac{b_t}{R_t}$, to the firm owner as a dividend.

3.2 Derivation of the borrowing constraint

In this subsection, we describe the events that follow a counterfactual default on σ_t, and derive the borrowing constraint (4) as the no-default condition. The argument here is similar to that of Jermann and Quadrini (2012).

As described in the previous subsection, the firm owes inter-period debt $\frac{b_t}{R_t}$ and intra-period debt σ_t at the end of period t, where b_t is to be repaid in period $t+1$ and σ_t is to

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[9] More accurately, the firm owes $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}$ to the bank. Hence, the firm has to pay this amount to the bank, whereas it obtains a transfer from the government as a tax advantage, amounting to $\tau r_{t-1}\frac{b_{t-1}}{R_{t-1}}$. Thus, the net payment by the firm is $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \tau r_{t-1}\frac{b_{t-1}}{R_{t-1}} = b_{t-1}$.

be repaid in period t. At the end of period t, the firm has the chance to default on σ_t.

Now, we consider what would happen if the firm defaults on σ_t. Once the firm defaults, the bank unilaterally seizes a part of the firm's revenue, $\phi f(\sigma_t, x_t)$, where $0 \leq \phi < 1$.[10] Then, the firm and bank renegotiate over the conditions for the firm to continue to operate. At this stage, the bank has acquired the right to liquidate the firm. If the bank decides to liquidate the firm, it obtains value S_t with probability ξ, whereas the firm is destroyed with probability $1-\xi$. This is because when the bank chooses liquidation, it successfully operates the firm by itself and recovers value S_t with probability ξ. Thus, the expected value that the bank can obtain by liquidation is ξS_t. In contrast, if the bank decides to allow the firm to continue to operate, it can recover its inter-period debt in the next period, the present value of which is $\frac{b_t}{R_t}$. The agreement of renegotiation depends on whether ξS_t is larger or smaller than $\frac{b_t}{R_t}$.

- **Case where $\xi S_t > \frac{b_t}{R_t}$:** The firm has to make a payment that leaves the bank indifferent between liquidation and allowing the firm to continue to operate.[11] Thus, the firm has to make payment $\xi S_t - \frac{b_t}{R_t}$ and promise to pay $(1+r_t)\frac{b_t}{R_t}$ at the beginning of the next period.Therefore, the ex-post default value for the firm is

$$(1-\phi)f(\sigma_t, x_t) - b_{t-1} + \frac{b_t}{R_t} - \left\{\xi S(x_t) - \frac{b_t}{R_t}\right\} + \mathbb{E}_t\left[\frac{m_{t+1}}{m_t}V(b_t, x_{t+1})\right].$$

- **Case where $\xi S_t \leq \frac{b_t}{R_t}$:** In this case, the optimal choice for the bank is to wait until the next period when $(1+r_t)\frac{b_t}{R_t}$ is due. In period t, the bank receives no further payments. Thus, the ex-post default value is

$$(1-\phi)f(\sigma_t, x_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t\left[\frac{m_{t+1}}{m_t}V(b_t, x_{t+1})\right].$$

Therefore, the default value is expressed as

$$(1-\phi)f(\sigma_t, x_t) - b_{t-1} + \frac{b_t}{R_t} - \max\left\{\xi S(x_t) - \frac{b_t}{R_t},\ 0\right\} + \mathbb{E}_t\left[\frac{m_{t+1}}{m_t}V(b_t, x_{t+1})\right].$$

---

[10] Because the firm has paid $b_{t-1} - \frac{b_t}{R_t}$, the remaining value of resources it possesses is $f(\sigma_t, x_t) - b_{t-1} + \frac{b_t}{R_t}$, after defaulting on σ_t. Thus, if the bank were to seize $\phi f(\sigma, x)$ from the remaining output only, then the seizure should be feasible only when

$$b_{t-1} - \frac{b_t}{R_t} \leq (1-\phi)f(\sigma_t, x_t). \tag{10}$$

This condition should be a constraint for the firm and the bank if the bank were to seize resources from the remaining output. However, we assume that the bank takes $\phi f(\sigma, x)$ from the firm-owner's pocket, not only from the remaining output of the firm. Because the firm owner does not have liquid assets in hand, the bank cannot collect $\phi f(\sigma, x)$ immediately, whereas it can recover this present value from the firm-owner's illiquid assets within some period. Thus, the bank seizure is not constrained by (10).

[11] The participation constraint for the bank should be binding because we assumed that the firm has all the bargaining power in the renegotiation. Jermann and Quadrini (2012) posited the same assumption.

Enforcement requires that the value of not defaulting is no smaller than the value of defaulting, that is,

$$f(\sigma_t, x_t) - b_{t-1} + \frac{b_t}{R_t} - \sigma_t \geq (1-\phi)f(\sigma_t, x_t) - b_{t-1} + \frac{b_t}{R_t} - \max\left\{\xi S(x_t) - \frac{b_t}{R_t},\ 0\right\},$$

which can be rearranged as (4).

3.3 Equilibrium debt dynamics

In this subsection, for a given variable q_t, we denote the variables in the previous period, in the current period, and in the next period by q_{-1}, q, and q_{+1}, respectively. Here, we characterize the equilibrium path that solves (6), taking $S(x)$ as given exogenously. We prove the existence of the equilibrium that satisfies (7) in Section 3.4. First, we show the existence of the solution to (6), given exogenous $S(x)$. Let G be the following closed, bounded, and convex set of non-negative continuous functions, $S(x_t)$:

$$G = \{S(x) | 0 \leq S(x) \leq \bar{\omega},\ x \in \Lambda,\ S(x) \in C(\Lambda)\}.$$

Proposition 1. *Let $S(x)$ $(\in G)$ be given exogenously. There exists a solution $V(b, x; S)$ to the Bellman equation* (6)*, and $V(b, x; S)$ is continuous in (b, x).*

The proof is omitted because this proposition follows directly from Theorem 9.6 in Stokey and Lucas with Prescott (1989).[12]

In what follows, we assume the following two assumptions hold.

Assumption 1. The parameters satisfy the following condition:

$$\xi\bar{\omega} < \frac{(1-\phi)\underline{\mathrm{f}}_z}{r_{\max}},$$

where $\underline{\mathrm{f}}_z \equiv \inf_{x\in\Lambda} f(\sigma_z(x), x)$ and $r_{\max} \equiv \sup_{x\in\Lambda} r(x)$.

This assumption implies that once the amount of debt $\frac{b_t}{R_t}$ becomes smaller than $\xi\bar{\omega}$, it can reach the constrained-efficient level $b^{ce}(x)$ (defined below) within a finite number of periods. This is because the firm can repay at least $\frac{(1-\phi)\underline{\mathrm{f}}_z}{r_{\max}}$ by paying all income to the bank in all future periods. Define

$$\mu^{ce}(x) \equiv \frac{\tau r(x)}{1 + r(x)}.$$

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[12] To see that Theorem 9.6 in Stokey and Lucas with Pescott (1989) is applicable to the problem in (6), it is useful to change the variables by $\tilde{m}_t \equiv \beta^{-t} m_t$ and $\tilde{V}(b_{t-1}, x_t) \equiv \tilde{m}_t V(b_{t-1}, x_t)$. The Bellman equation (6) can be rewritten as

$$\tilde{V}(b_{t-1}, x_t) = \max_{b_t, \sigma_t, \pi_t} \tilde{m}_t \pi_t + \beta \mathbb{E}_t \left[\tilde{V}(b_t, x_{t+1})\right],$$

subject to the same constraints. Assumptions 9.4–9.7 in Stokey and Lucas with Prescott (1989) are clearly satisfied in this problem, which allows Theorem 9.6 to apply.

Then, $\sigma^{ce}(x)$ is defined as the solution to

$$\frac{\partial}{\partial\sigma} f(\sigma, x) = \frac{1+\mu^{ce}(x)}{1+\phi\mu^{ce}(x)}.$$

Define $b^{ce}(x; S)$, where $S = S(x)$, by

$$\frac{b^{ce}(x;S)}{R(x)} \equiv \phi f(\sigma^{ce}(x), x) - \sigma^{ce}(x) + \xi S(x).$$

Assumption 2. The parameters satisfy the following conditions:

$$\sup_{x\in\Lambda} \phi f(\sigma^{ce}(x), x) - \sigma^{ce}(x) < 0,$$

such that $\frac{b^{ce}(x;S)}{R(x)} < \xi S(x)$ for all x, and

$$\inf_{x,x_{-1}\in\Lambda, S\in G} \left[f(\sigma^{ce}(x), x) - \sigma^{ce}(x) - b^{ce}(x_{-1}; S) + \frac{b^{ce}(x;S)}{R(x)} \right] > 0.$$

The first condition of this assumption is satisfied for a small τ.[13] The second condition of this assumption implies that once the economy enters the constrained-efficient equilibrium, it stays there forever. Constrained efficiency is defined and explained in Section 3.3.2. We define the total surplus of the match of the firm and bank, $W(b_{-1}, x)$, as

$$W(b_{-1}, x) = \frac{1+r_{-1}}{R_{-1}} b_{-1} + V(b_{-1}, x).$$

Note that the bank receives $\frac{1+r_{-1}}{R_{-1}} b_{-1}$ from the firm, while the net payment for the firm is b_{-1}, because the government gives tax advantage $\frac{\tau r_{-1}}{R_{-1}} b_{-1}$ to the firm. Thus, the firm's problem is equivalent to maximizing $W(b_{-1}, x)$, given b_{-1}. Function $W(b_{-1}, x)$ satisfies the following dynamic programming equation:

$$\begin{aligned} W(b_{-1}, x) = & \max_{\sigma, b} \frac{r_{-1}\tau}{R_{-1}} b_{-1} + f(\sigma, x) - \sigma + \mathbb{E}\left[\frac{m_{+1}}{m} W(b, x_{+1})\right], && (11) \\ \text{subject to } \; & f(\sigma, x) - \sigma - b_{-1} + \frac{b}{R} \geq 0, && (12) \\ & \sigma \leq \phi f(\sigma, x) + \max\left\{\xi S(x) - \frac{b}{R},\ 0\right\}, && (13) \\ & b \leq b_z(x). && (14) \end{aligned}$$

By denoting the Lagrange multipliers for (12), (13), and (14) as λ_π, μ, and ν, respectively, the FOCs and envelope condition are

$$\frac{\partial}{\partial\sigma} f(\sigma, x) = \frac{1+\frac{\mu}{1+\lambda_\pi}}{1+\phi\frac{\mu}{1+\lambda_\pi}}, \tag{15}$$

$$\begin{cases} R\mathbb{E}[\frac{m_{+1}}{m}\frac{\partial}{\partial b}W(b, x_{+1})] + \lambda_\pi - \nu - \mu = 0, & \text{if } \xi S(x) \geq \frac{b}{R}, \\ R\mathbb{E}[\frac{m_{+1}}{m}\frac{\partial}{\partial b}W(b, x_{+1})] + \lambda_\pi - \nu = 0, & \text{if } \xi S(x) < \frac{b}{R}, \end{cases} \tag{16}$$

$$\frac{\partial}{\partial b} W(b_{-1}, x) = \frac{r_{-1}\tau}{R_{-1}} - \lambda_\pi. \tag{17}$$

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[13] In the steady state, where the state is time-invariant, this condition is equivalent to $\frac{\tau r}{1+r} < \frac{\eta-\phi}{(1-\eta)\phi}$.

The equilibrium values of the five endogenous variables $\{\sigma, b, \lambda_\pi, \mu, \nu\}$, and $\frac{\partial}{\partial b}W(b_{-1}, x)$ are determined by the constraints (12)–(14) and the conditions (15)–(17).

3.3.1 Special case with $\tau = 0$

In the special case where $\tau = 0$, the firm is indifferent between debt and equity as a tool for funding, as long as the amount of debt is sufficiently small that the borrowing constraint is non-binding. When $\tau = 0$, the variables satisfy $\sigma^{ce}(x) = \sigma^*(x)$ and $b^{ce}(x) = b^*(x)$ in Assumption 2, which implies that once the economy enters the first-best equilibrium, where $\{\sigma_t, b_t\} = \{\sigma^*(x_t),\ b^*(x_t)\}$, it stays there forever: $\{\sigma_{t+j}, b_{t+j}\} = \{\sigma^*(x_{t+j}),\ b^*(x_{t+j})\}$, for all $j \geq 1$. This is because for any $b^*(x_{-1})$, the firm can choose $\sigma = \sigma^*(x)$ because π can be non-negative when it chooses $\sigma = \sigma^*(x)$ for all $x \in \Lambda(x_{-1})$.

We define $W^*(x)$, $b^*(x; S)$, $B^*(x; S)$, $B_z(x; S)$, and $\bar{B}_z(x; S)$ by

$$\begin{aligned}
W^*(x) &\equiv f(\sigma^*(x), x) - \sigma^*(x) + \mathbb{E}\left[\frac{m_{+1}}{m} W^*(x_{+1})\right], \\
\frac{b^*(x; S)}{R(x)} &\equiv \phi f(\sigma^*(x), x) - \sigma^*(x) + \xi S(x), \\
B^*(x; S) &\equiv f(\sigma^*(x), x) - \sigma^*(x) + \frac{b^*(x; S)}{R(x)}, \\
B_z(x; S) &\equiv f(\sigma_z(x), x) - \sigma_z(x) + \xi S(x), \\
\bar{B}_z(x) &\equiv f(\sigma_z(x), x) - \sigma_z(x) + \frac{b_z(x)}{R(x)}.
\end{aligned}$$

The meanings of the above variables are as follows: $W^*(x)$ is the first-best value of the match of the bank and firm, in the case where $\tau = 0$; $b^*(x_t)$ is the maximum amount of debt b_t that is feasible in the first-best equilibrium, where $\sigma_t = \sigma^*(x_t)$; $B^*(x_t; S)$ is the maximum amount of debt b_{t-1} repayable in the first-best equilibrium, where $\sigma_t = \sigma^*(x_t)$; $B_z(x_t; S)$ is the minimum amount of debt b_{t-1} that makes $\frac{b_t}{R_t} \geq \xi S(x_t)$; $\bar{B}_z(x_t)$ is the maximum amount of debt b_{t-1} repayable with certainty, given that the current state in period t is x_t. A capital B is used to represent the threshold of debt.

In the case where $\tau = 0$, the equilibrium dynamics are qualitatively the same as those in Albuquerque and Hopenhayn (2004). We can show the following lemma, which is equivalent to Lemma 1 in Albuquerque and Hopenhayn (2004). The difference is that the debt in our model is not state-contingent, while it is state-contingent in their model.

Lemma 2. *For a given state x,*

(i) $W(b_{-1}, x)$ is weakly decreasing in b_{-1}.

(ii) For all $b_{-1} \leq B^*(x; S)$, $W(b_{-1}, x) = W^*(x)$.

(iii) For all $b_{-1} > B^*(x; S)$, $W(b_{-1}, x) < W^*(x)$ and $\pi = 0$.

The proof is given in Appendix A. This lemma implies that if $b_{-1} > B^*(x)$, then the firm repays as much debt as possible by setting the dividend to zero: $\pi = 0$. This firm's behavior is the same as that in Albuquerque and Hopenhayn (2004). If $b_{-1} > B_z(x; S)$, then $\sigma = \sigma_z(x)$, where $\sigma_z(x)$ is the solution to $\sigma = \phi f(\sigma, x)$. Now, we turn to the general case where $\tau > 0$.

3.3.2 Case with $\tau > 0$

The features of the equilibrium path are characterized by the initial amount of inter-period debt b_{-1}. We define

$$B^{ce}(x; S) \equiv f(\sigma^{ce}(x), x) - \sigma^{ce}(x) + \frac{b^{ce}(x; S)}{R(x)}.$$

Constrained-efficient equilibrium with small debt: We focus on the case where initial debt b_{-1} is sufficiently small (i.e., $b_{-1} < B^{ce}(x; S)$). In this case, the borrowing constraint is written as

$$\sigma \leq \phi f(\sigma, x) + \xi S(x) - \frac{b}{R}. \tag{18}$$

When $\mu > 0$, the FOCs imply that $\frac{\partial}{\partial k} F(A, k, l) > r^K$ and $\frac{\partial}{\partial l} F(A, k, l) > w$. It immediately follows that inputs k and l are inefficiently smaller than their efficient levels, k^* and l^*, which are given by $\frac{\partial}{\partial k} F(A, k^*, l^*) = r^K$ and $\frac{\partial}{\partial l} F(A, k^*, l^*) = w$, respectively. Note that the borrowing constraint is always binding (i.e., $\mu > 0$) because the firm borrows inter-temporal debt to exploit the tax advantage. We call the equilibrium *constrained-efficient* if $\lambda_{\pi,t} = \nu_t = 0$ for all t. We assume and justify later that $\lambda_\pi = \lambda_{\pi,+1} = \nu = 0$, when $b_{-1} < B^{ce}(x; S)$. In this case, conditions (15)–(17) imply $\mathbb{E}[\frac{m_{+1}}{m}]\frac{r\tau}{R} = \frac{\mu}{R}$, which can be rewritten as $\mu = \mu^{ce}(x)$. Therefore, the tightness of the borrowing constraint (13) (or (18)) in the constrained-efficient equilibrium, $\mu^{ce}(x)$, is not dependent on b_{-1}, and is decided solely by the tax advantage. Therefore, the value of σ in the constrained-efficient equilibrium, $\sigma^{ce}(x)$, is also independent of b_{-1}. The value of b should be $b^{ce}(x; S)$. Note that Assumption 2 guarantees that once the economy enters the constrained-efficient equilibrium, it stays there forever. This is because, for any $b^{ce}(x_{-1})$, the firm can choose $\sigma = \sigma^{ce}(x)$ because π can be non-negative when it chooses $\sigma = \sigma^{ce}(x)$ for all $x \in \Lambda(x_{-1})$. Thus our assumption that $\lambda_\pi = \lambda_{\pi,+1} = \nu = 0$ is justified.

Medium-sized debt: Consider the case where initial debt b_{-1} is medium-sized (i.e., $B^{ce}(x; S) \leq b_{-1} < B_z(x; S)$). In this case, the borrowing constraint is still (18). We focus on the cases where the parameter values are chosen such that $\nu = 0$ in equilibrium when b_{-1} is medium-sized. Then, we have the following lemma.

Lemma 3. *Consider the case where b_{-1} is medium-sized and $\sigma < \sigma^{ce}(x)$ in equilibrium. Then, $\lambda_\pi > 0$ and the dividend is equal to zero, $\pi = 0$, in equilibrium.*

Proof. Proof is by contradiction. Suppose that $\lambda_\pi = 0$. Then, conditions (16) and (17) imply that $\mu = R\mathbb{E}[\frac{m_{+1}}{m}(\frac{r\tau}{R} - \lambda_{\pi+1})] \leq \frac{r\tau}{1+r} = \mu^{ce}$. (15) implies that $\sigma \geq \sigma^{ce}(x)$. This is a contradiction. Thus, $\lambda_\pi = 0$ cannot hold in equilibrium. Hence, $\lambda_\pi > 0$ and $\pi = 0$ in equilibrium. □

The values of $\{k, l, b, \mu\}$ are determined by the budget constraint (3) with $\pi = 0$, the borrowing constraint (18), the FOC (15), and

$$\frac{k}{l} = \left(\frac{\alpha}{1-\alpha}\right)\frac{w}{r^K},$$

where the last equation is derived from the conditions for k and l to maximize $F(A, k, l)$ subject to $r^K k + wl \leq \sigma$. Given that the outstanding debt b_{-1} is medium-sized, the firm repays as much debt as possible by setting $\pi = 0$, and eventually returns to the constrained-efficient equilibrium. Assumption 1 implies that it takes a finite number of periods to return to the constrained-efficient equilibrium. These features are qualitatively the same as those of debt repayment in Albuquerque and Hopenhayn (2004).

Large debt: If $b_{-1} \geq B_z(x)$, the borrowing constraint becomes

$$\sigma \leq \phi f(\sigma, x). \tag{19}$$

Then, the FOC (16) implies

$$\mathbb{E}\left[\frac{m_{+1}}{m}\frac{\partial}{\partial b}W(b, x_{+1})\right] = \frac{\nu - \lambda_\pi}{R}.$$

It immediately follows that

- If $\lambda_\pi > 0$, then $\pi = 0$, $b = R\{b_{-1} - f(\sigma_z(x), x) + \sigma_z(x)\}$, and $\nu = 0$.
- If $\nu > 0$, debt jumps to the upper limit (i.e., $b = b_z(x)$), $\lambda_\pi = 0$, and $\pi > 0$.

Note that although the FOCs should be satisfied at the solution, the choice between $b = R\{b_{-1} - f(\sigma_z(x), x) + \sigma_z(x)\}$ and $b = b_z(x)$ cannot be determined by the FOCs alone because they are the local maxima. The firm compares the total surplus when $b = R\{b_{-1} - f(\sigma_z(x), x) + \sigma_z(x)\}$ and that when $b = b_z(x)$, and then chooses the higher surplus. Define $W^{(1)}(b_{-1}, x)$ and $W^{(2)}(b_{-1}, x)$ by

$$W^{(1)}(b_{-1}, x) = \frac{\tau r_{-1}}{R_{-1}}b_{-1} + f(\sigma_z(x), x) - \sigma_z(x) + \mathbb{E}\left[\frac{m_{+1}}{m}W(b_z(x), x_{+1})\right],$$
$$W^{(2)}(b_{-1}, x) = \frac{\tau r_{-1}}{R_{-1}}b_{-1} + f(\sigma_z(x), x) - \sigma_z(x) + \mathbb{E}\left[\frac{m_{+1}}{m}W(R\{b_{-1} - f(\sigma_z, x) + \sigma_z\}, x_{+1})\right].$$

Here, $W^{(1)}(b_{-1}, x)$ is the total surplus when the firm chooses $b = b_z(x)$ and $W^{(2)}(b_{-1}, x)$ is that when the firm chooses $b = R\{b_{-1} - f(\sigma_z(x), x) + \sigma_z(x)\}$. Then, $W(b_{-1}, x)$ is given by

$$W(b_{-1}, x) = \max\ \{W^{(1)}(b_{-1}, x),\ W^{(2)}(b_{-1}, x)\}.$$

We assume that the values of the parameters are chosen such that

$$W^{(1)}(B_z(x;S),x) < W^{(2)}(B_z(x;S),x), \qquad \text{for all } x. \tag{20}$$

By definition of $\bar{B}_z(x)$, the following equality holds:

$$W^{(1)}(\bar{B}_z(x),x) = W^{(2)}(\bar{B}_z(x),x), \qquad \text{for all } x. \tag{21}$$

Then, the following lemma holds.

Lemma 4. *There exist* $\underline{B}_c(x;S)$ *and* $\bar{B}_c(x;S)$*, such that* $B_z(x;S) < \underline{B}_c(x;S) \leq \bar{B}_c(x;S) \leq \bar{B}_z(x)$ *and*

- *If* $b_{-1} \in [B_z(x;S), \underline{B}_c(x;S))$*, then* $W^{(1)}(b_{-1},x) < W^{(2)}(b_{-1},x)$*,* $\pi = 0$*, and* $b = R\{b_{-1} - f(\sigma_z(x),x) + \sigma_z(x)\}$*.*
- *If* $b_{-1} \in [\bar{B}_c(x;S), \bar{B}_z(x)]$*, then* $W^{(1)}(b_{-1},x) \geq W^{(2)}(b_{-1},x)$ *and* $b = b_z(x)$*.*

Proof. The lemma follows immediately from the continuity of $W^{(1)}(b_{-1},x)$ and $W^{(2)}(b_{-1},x)$, together with (20) and (21). ☐

In the general case with stochastic shocks, we can only say that $\bar{B}_c(x;S) \leq \bar{B}_z(x)$, while in the deterministic case, where x_t evolves deterministically, it is shown that $\bar{B}_{c,t} < \bar{B}_{z,t}$. Thus, the firm intentionally increases borrowing to $b_{z,t}$ when outstanding debt b_{t-1} is sufficiently large, such that $\bar{B}_{c,t} \leq b_{t-1} < \bar{B}_{z,t}$.

Debt-ridden firms: A *debt-ridden firm* is defined as a firm with outstanding debt $b_{t-1} = b_z(x_{t-1})$. From the definition of $b_z(x_t)$, the borrowing constraint is (19) for all t in the case where x_t evolves deterministically. In the case where x_t evolves stochastically, there may be a nonzero probability with which the firm repays its debt sufficiently and eventually returns to the constrained-efficient equilibrium.

3.4 Existence of an equilibrium

We show that there exists an equilibrium in which the firm solves (6) and the equilibrium condition (7) is satisfied. We need the following restriction on the parameters.

Assumption 3. The values of the parameters satisfy

$$\frac{\eta}{2-\eta} < \phi,$$

and

$$\inf_{x \in \Lambda} (1-\phi) f(\sigma_z(x),x) > \nabla,$$

where ∇ is defined by

$$\nabla = \sup_{x \in \Lambda, x_{+1} \in \Lambda(x), S \in G} \left| B^{ce}(x;S) - \frac{B^{ce}(x_{+1};S)}{R(x)} \right|.$$

Note that this assumption is always satisfied in the case where the state and prices do not change over time. In this case, $\inf_{x\in\Lambda}(1-\phi)f(\sigma_z(x),x) > \nabla$ can be rewritten as $(1-\phi)f(\sigma_z) = rb_z > \frac{R-1}{R}B^{ce}$, which is obviously satisfied because $b_z > B^{ce}$ from (9) and $1 < R < 1+r$.

Proposition 5. *Let Assumptions 1–3 be satisfied. There exists an equilibrium in which the firm solves the optimization problem* (6)*, taking* $S(x)$ *as given, and* (7) *is satisfied.*

The proof is given in Appendix B. We have established the existence of an equilibrium, but not the uniqueness. It may not be possible to show such uniqueness analytically. However, in the deterministic case, where state x_t evolves deterministically, we can use a numerical simulation to show that the equilibrium is unique for a range of parameter values that we consider to be relevant.

3.5 A deterministic case

In this subsection, we focus on the equilibrium that is shown to exist in Proposition 5, where $S(x)$ is given endogenously by (7). In the case where state x_t evolves deterministically, the state of nature can be indicated by time. Thus, in this subsection, we use the time subscript t to represent state x_t. We also use prime to indicate differentiation. The firm's problem is

$$\begin{aligned}
W_t(b_{t-1}) = &\max_{\sigma_t, b_t} \frac{r_{t-1}\tau}{R_{t-1}} b_{t-1} + f_t(\sigma_t) - \sigma_t + \frac{m_{t+1}}{m_t} W_{t+1}(b_t), \\
\text{subject to} \quad & f_t(\sigma_t) - \sigma_t - b_{t-1} + \frac{b_t}{R_t} \geq 0, \\
& \sigma_t \leq \phi f_t(\sigma_t) + \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}, \\
& b_t \leq b_{z,t}.
\end{aligned}$$

The FOCs and envelope condition are

$$\begin{aligned}
& f_t'(\sigma_t) = \frac{1 + \frac{\mu_t}{1+\lambda_{\pi,t}}}{1 + \phi\frac{\mu_t}{1+\lambda_{\pi,t}}}, \\
& \begin{cases} \frac{m_{t+1}}{m_t} W_{t+1}'(b_t) + \frac{\lambda_{\pi,t} - \nu_t - \mu_t}{R_t} = 0, & \text{if } \xi S_t \geq \frac{b_t}{R_t}, \\ \frac{m_{t+1}}{m_t} W_{t+1}'(b_t) + \frac{\lambda_{\pi,t} - \nu_t}{R_t} = 0, & \text{if } \xi S_t < \frac{b_t}{R_t}, \end{cases} \\
& W_t'(b_{t-1}) = \frac{\tau r_{t-1}}{R_{t-1}} - \lambda_{\pi,t}.
\end{aligned}$$

Permanent inefficiency of a debt-ridden firm: If a firm is debt-ridden (i.e., $b_{t-1} = b_{z,t-1}$), inefficiency continues permanently because its debt does not decrease, even if it sets $\pi_t = 0$ and repays as much debt as possible for all t. The following lemma is established immediately from the definition of $b_{z,t}$ in the deterministic economy (see (8)).

Lemma 6. *Once $b_t = b_{z,t}$, then $b_{t+j} = b_{z,t+j}$ and $\sigma_{t+j} = \sigma_{z,t+j}$, for all $j \geq 0$.*

This lemma means that once a firm becomes debt-ridden, it continues to be debt-ridden and production is inefficient forever. Note that the value of a debt-ridden firm is zero, because $\pi_t = 0$ for all t, that is,

$$V_t(b_{z,t-1}) = 0 \quad \text{and} \quad W_t(b_{z,t-1}) = \frac{1 + r_{t-1}}{R_{t-1}} b_{z,t-1}.$$

We define W_t^*, $\frac{b_t^*}{R_t}$, B_t^*, $B_{z,t}$, $\bar{B}_{z,t}$, and B_t^{ce} using the same definitions as their counterparts in the stochastic case. Note that $\bar{B}_{z,t} = b_{z,t}$.

3.5.1 Special case where $\tau = 0$

In this case, the optimal equilibrium is characterized by $\lambda_{\pi,t} = \nu_t = \mu_t = 0$ and $f_t'(\sigma_t) = 1$. The FOCs and envelope condition imply the following lemma.

Lemma 7. *If $\lambda_{\pi,t} = 0$, the economy stays in the optimal equilibrium from t onward; that is, $\lambda_{\pi,t+j} = \nu_{t+j} = \mu_{t+j} = 0$ and $f_{t+j}'(\sigma_{t+j}) = 1$, for all $j \geq 0$.*

Lemma 8. *(i) $W_t(b)$ is weakly decreasing in b.*

(ii) For all $b_{t-1} \leq B_t^*$, the economy is in the optimal equilibrium (i.e., $W_t(b_{t-1}) = W_t^*$).

(iii) For all $b_{t-1} > B_t^*$, $W_t(b_{t-1}) < W_t^*$ and $\pi_t = 0$.

The above item (i) follows from $W_t'(b_{t-1}) = -\lambda_{\pi,t}$ and $\lambda_{\pi,t} \geq 0$. This lemma implies that if $b_{t-1} \in (B_t^*, \bar{B}_{z,t})$, where $\bar{B}_{z,t} = b_{z,t-1}$, production is inefficient ($f_t'(\sigma_t) > 1$), and the firm sets $\pi_t = 0$ and repays as much debt as possible to reduce the amount of remaining debt b_t. When b_{t-1} becomes small and satisfies $b_{t-1} \leq B_t^*$, production becomes efficient ($f_t'(\sigma_t) = 1$) and the firm is indifferent about the amount of debt because its value no longer changes by repaying debt. If a firm is debt-ridden (i.e., $b_{t-1} = b_{z,t-1}$), inefficiency continues permanently (see Lemma 6). Thus, in the case where $\tau = 0$, the firm repays debt and returns to normal eventually, as long as the initial debt, b_{-1}, is strictly smaller than the upper limit, $b_{z,-1}$. However, it remains debt-ridden and continues inefficient production permanently if the initial debt is $b_{z,-1}$.

3.5.2 Case with $\tau > 0$

In the case where $\tau > 0$, the constrained-efficient equilibrium, characterized by $\lambda_{\pi,t} = \nu_t = 0$, $\pi_t > 0$, $\mu_t^{ce} = \frac{\tau r_t}{1+r_t}$, $f'(\sigma_t^{ce}) = \frac{1+\mu_t^{ce}}{1+\phi\mu_t^{ce}}$, and $b_t = b_t^{ce}$ for all t, is attained when the debt is sufficiently small (i.e., $b_{t-1} < B_t^{ce}$). For medium-sized debt (i.e., $b_{t-1} \in [B_t^{ce},\ B_{z,t})$), the firm sets the dividend to zero ($\pi_t = 0$) and repays as much debt as possible. Assumption 1 guarantees that it goes to the constrained-efficient equilibrium within finite periods.

This result is the same as in Albuquerque and Hopenhayn (2004). For large debt (i.e., $b_{t-1} \in [B_{z,t},\ \bar{B}_{z,t}]$), the borrowing constraint becomes (19). Note that $\bar{B}_{z,t} = b_{z,t-1}$ in the deterministic economy. When debt is large, the following two cases exist in equilibrium:

(i) $\nu_t = 0$: Either $\lambda_{\pi,t} > 0$ or $\lambda_{\pi,t} = 0$. If $\lambda_{\pi,t} > 0$, then $b_{t+1} < b_t$, and the FOC and envelope condition imply that $\lambda_{\pi,t+j} > 0$ for $j = 1, 2, \cdots$ as long as the borrowing constraint at $t + j$ is (19). If $\lambda_{\pi,t} = 0$, then the FOC and envelope condition imply that $\lambda_{\pi,t+1} > 0$ and, thus, that $\lambda_{\pi,t+j} > 0$ for $j = 2, 3, \cdots$, as long as the borrowing constraint at $t+j$ is (19). In any case, the firm eventually returns to the constrained-efficient equilibrium by setting $\pi_t = 0$ to repay as much debt as possible.

(ii) $\nu_t > 0$: In this case, $b_t = b_{z,t}$, that is, the firm intentionally chooses to increase borrowing and become debt-ridden. Once the firm becomes debt-ridden, it stays there forever (Lemma 6).

For $b_{t-1} \in (B_{z,t},\ \bar{B}_{z,t})$, where $\bar{B}_{z,t} = b_{z,t-1}$, the equilibrium path is determined as follows: the firm compares the total surplus when $b_t = R_t\{b_{t-1} - f_t(\sigma_{z,t}) + \sigma_{z,t}\}$ and that when $b_t = b_{z,t}$, and then chooses the higher surplus. For $B_{z,t} \le b_{t-1} < b_{z,t-1}$, the Bellman equation reduces to

$$W_t(b_{t-1}) = \max\{W_t^{(1)}(b_{t-1}),\ W_t^{(2)}(b_{t-1})\},$$

$$\text{where} \quad W_t^{(1)}(b_{t-1}) = \frac{\tau r_{t-1}}{R_{t-1}} b_{t-1} + f_t(\sigma_{z,t}) - \sigma_{z,t} + \frac{1}{1+r_t} W_{t+1}(b_{z,t}),$$

$$W_t^{(2)}(b_{t-1}) = \frac{\tau r_{t-1}}{R_{t-1}} b_{t-1} + f_t(\sigma_{z,t}) - \sigma_{z,t} + \frac{1}{1+r_t} W_{t+1}(R_t\{b_{t-1} - f_t(\sigma_{z,t}) + \sigma_{z,t}\}).$$

We show that there exists $\bar{B}_{c,t}$, such that $\bar{B}_{c,t} < b_{z,t-1}$ and $W_t^{(1)}(b_{t-1}) > W_t^{(2)}(b_{t-1})$, for $b_{t-1} \in (\bar{B}_{c,t}, b_{z,t-1})$. To prove this, we need the following restriction on the parameter values.

$$\inf_t \frac{1+r_t}{R_t} > 1. \tag{22}$$

Proposition 9. *In the deterministic case, there exist* $\underline{B}_{c,t}$ *and* $\bar{B}_{c,t}$*, where* $B_{z,t} \le \underline{B}_{c,t} \le \bar{B}_{c,t} < b_{z,t-1}$*, such that*

- *If* $B_{z,t} \le b_{t-1} < \underline{B}_{c,t}$*, then* $W_t^{(1)}(b_{t-1}) < W_t^{(2)}(b_{t-1})$*. Here, the firm sets the dividend to zero (i.e.,* $\pi_t = 0$*) to repay as much debt as possible, and goes to the constrained-efficient equilibrium in finite periods.*
- *If* $b_{t-1} > \bar{B}_{c,t}$*, then* $W_t^{(1)}(b_{t-1}) > W_t^{(2)}(b_{t-1})$*. In this case, the firm intentionally increases debt to the upper limit (i.e.,* $b_t = b_{z,t}$*), and remains debt-ridden forever.*

The proof is given in Appendix C. Note that the threshold $\bar{B}_{c,t}$ is strictly smaller than $b_{z,t-1}$. It is likely that

$$\underline{B}_{c,t} = \bar{B}_{c,t}\ (\equiv B_{c,t})$$

for a wide range of parameter values in our numerical experiment.

Why does a heavily indebted firm choose to become debt-ridden? The above proposition implies that there exists $b_{t-1} \in [B_{c,t}, b_{z,t-1})$, such that the firm that owes b_{t-1} intentionally increases borrowing and becomes debt-ridden (i.e., $b_t = b_{z,t}$). Intuitively, suppose, for simplicity, that all prices are constant over time. Then, suppose the initial debt b is large, meaning that $B^S < b < b_z$, where B^S is the level of debt that satisfies $\frac{B^S}{R} = \xi S$. Then, it takes n periods to reduce the debt to B^S if the firm continues repaying as much as possible in every period. The value of n is uniquely determined as a function of b (i.e., $n = n(b)$), n is increasing in b, and $\lim_{b \uparrow b_z} n(b) = \infty$. Now, we can calculate the gain and loss of choosing to become debt-ridden at $t = 0$ in comparison with choosing to repay as much debt as possible. It is shown that the gain is proportional to $\frac{1}{R^n}$, whereas the loss is proportional to $\frac{1}{(1+r)^n}$, where $n = n(b)$. We know $1 + r > R$ because $\tau > 0$. Therefore, there exists n_c such that if $n > n_c$, then the gain of choosing to be debt-ridden exceeds the loss. We define B_c by $n_c = n(B_c)$. Then, the optimal choice for a firm with b ($> B_c$) is to become debt-ridden.

Our result that inefficiency owing to outstanding debt can continue indefinitely contrasts sharply with the findings in the existing literature on financial frictions. In standard models such as Carlstrom and Fuerst (1997) and Bernanke, Gertler and Gilchrist (1999), financial frictions have only temporary negative effects. In our model, a debt stock has a potentially indefinite negative effect on the output of the borrower.

3.5.3 Numerical example of the deterministic model

The numerical example of the deterministic case is shown in Figures 3 and 4. We assume that prices are invariant over time. The values of the parameters are chosen such that $\alpha = 0.3,\ \tau = 0.3,\ A = 0.885,\ w = 0.791,\ r = 0.02,\ r^K = 0.12,\ R = 1.014,\ \eta = 0.83,\ \phi = 0.71, and\ \xi = 0.75$. Figure 3 shows the response of the economy to a buildup of debt. Initially, the economy is in the steady state, where the level of debt is 3.95. At $t = 0$, debt b_{-1} suddenly increases to 4.026 because of an exogenous shock (e.g., the breakout of a financial crisis). The value of b_{-1} is chosen such that $B_z < b_{-1} < b_z$, where B_z and b_z are 4.016 and 4.73, respectively. The firm repays as much debt as possible, meaning that the dividend is zero for periods 0–3. Production is inefficient for periods 0–3 and returns to the constrained-efficient level in period 4. The borrowing constraint is $\sigma_t \leq \phi f(\sigma_t)$ for periods 0 and 1 and is $\sigma_t \leq \phi f(\sigma_t) + \xi S - \frac{b_t}{R}$ from period 2 onward, in this example. Note that $\sigma_0 = \sigma_1 = 0.163$ ($= \sigma_z$). In the case where the initial increase in debt b_{-1} is larger, the borrowing constraint continues to stay at $\sigma_t \leq \phi f(\sigma_t)$ for arbitrarily longer periods, thereby allowing inefficient production, $\sigma_t = \sigma_z$, to continue persistently. This example shows that $\frac{b_z}{R} = 4.66$ and $\xi S = 3.95$, meaning that $\xi S < \frac{b_z}{R}$.

Figure 4 shows the policy functions $b_t = b(b_{t-1})$ and $\sigma_t = \sigma(b_{t-1})$ and the value function

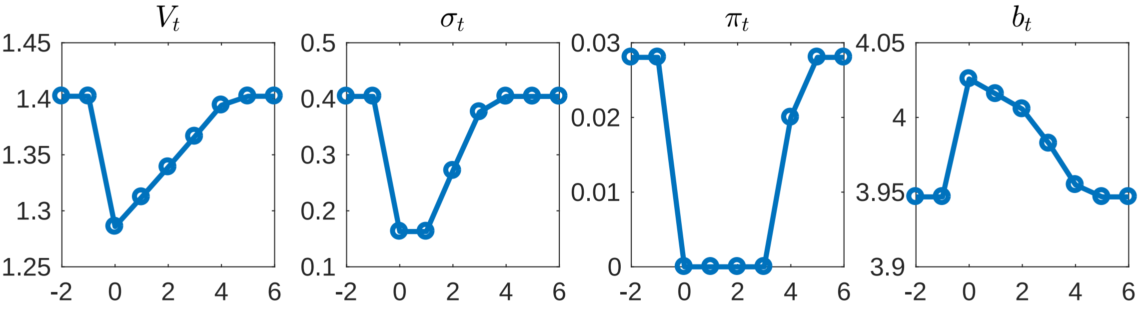

Figure 3: Responses to a buildup of debt

$V_t = V(b_{t-1})$. The policy functions have a kink at $b_{t-1} = B_z = (1-\phi)f(\sigma_z)+\xi S$, which is the boundary of the borrowing constraint between $\sigma_t \leq \phi f(\sigma_t)$ and $\sigma_t \leq \phi f(\sigma_t)+\xi S - \frac{b_t}{R}$. The policy function $b(b_{t-1})$ shows that debt decreases rapidly in the region where $b_{t-1} \leq B_z$, while the graph $b_t = b(b_{t-1})$ is close to the 45-degree line in the region where $b_{t-1} > B_z$. Thus, the speed of the decrease in debt is extremely slow. This figure indicates that the economy can suffer from extremely persistent inefficiency if $b_{t-1} > B_z$. Debt jumps to b_z for $b_{t-1} > B_c$. The policy function $\sigma(b_{t-1})$ shows that production becomes inefficient when the dividend is 0. Production becomes most inefficient (i.e., $\sigma_t = \sigma_z$) when $b_{t-1} \geq B_z$.

A note on the kinks and jumps in the policy function: As Figure 4 shows, the policy function $\sigma_t = \sigma(b_{t-1})$ has a kink at the boundary $b_{t-1} = B_z$. The reason why this kink appears is the structural change in the borrowing constraint at $b_{t-1} = B_z$, which seems similar to the mechanism of generating kinks in the models of a zero lower bound on nominal interest rates (Christiano and Fisher, 2000; Guerrieri and Iacoviello, 2015). In this numerical experiment, the policy function $\sigma_t = \sigma(b_{t-1})$ is continuous at $b_{t-1} = B_z$, even though it has a kink. However, it is continuous only because we assume that $\frac{\eta}{2-\eta} < \phi$ in our experiment. If $\phi < \frac{\eta}{2-\eta}$, the policy function is no longer continuous, and has a jump at $b_{t-1} = B_z$. The existence of the jump is explained as follows. The binding non-negativity condition $\pi_t = 0$ and borrowing constraint ($\sigma_t = \phi f(\sigma_t) + \xi S - \frac{b_t}{R}$) imply that σ_t solves the following equation:

$$b_{t-1} - \xi S = (1+\phi)f(\sigma_t) - 2\sigma_t,$$

which may have two solutions. For $b_{t-1} = B_z = (1-\phi)f(\sigma_z)+\xi S$, there exist two solutions, σ_1 and σ_2, such that $\sigma_1 = \sigma_z$ and $\sigma_2 > \sigma_1$ if $\phi < \frac{\eta}{2-\eta}$. If $\phi > \frac{\eta}{2-\eta}$, then $\sigma_2 < \sigma_1 = \sigma_z$, and if $\phi = \frac{\eta}{2-\eta}$, then $\sigma_2 = \sigma_1 = \sigma_z$. Therefore, if $\phi < \frac{\eta}{2-\eta}$, the policy function $\sigma(b_{t-1})$ jumps from σ_z to σ_2 ($> \sigma_z$) as b_{t-1} decreases slightly from B_z. Because continuity of $\sigma(b_{t-1})$ at $b_{t-1} = B_z$ is necessary to prove Proposition 5, we assume $\frac{\eta}{2-\eta} < \phi$.

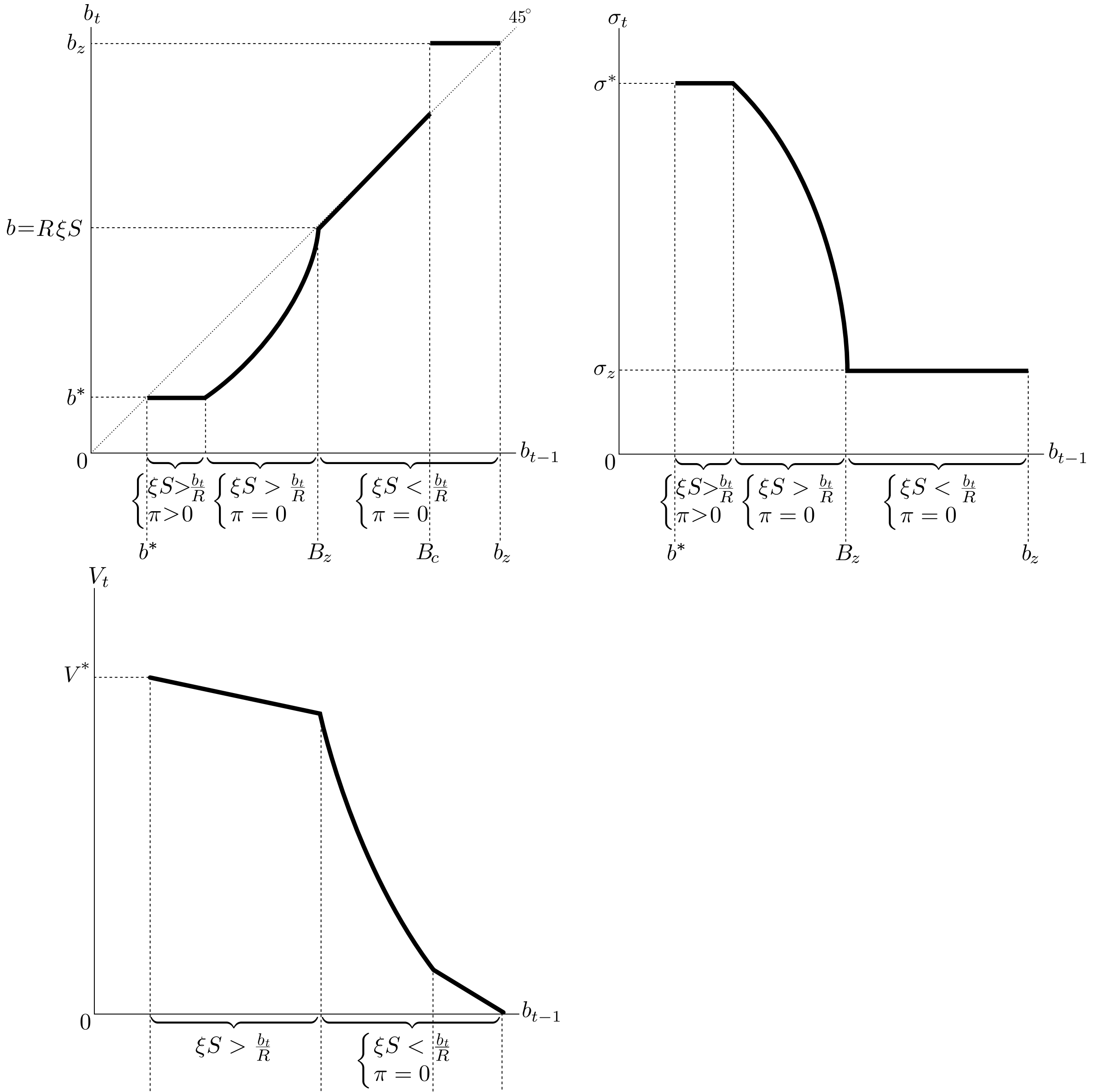

Figure 4: The policy functions and value function

4 Full model

In this section, we embed the partial equilibrium model of borrowing constraints into a general equilibrium model of endogenous growth. We consider a closed economy in which the final good is produced competitively from varieties of intermediate goods. The firms are monopolistic competitors and they produce their respective varieties of intermediate goods from the capital and labor inputs. This is a version of the expanding variety model, in which the new entry of firms increases aggregate productivity (Rivera-Batiz and Romer, 1991; Acemoglu, 2009). We follow Benassy (1998) in that labor is used to produce the intermediate goods, and only labor is used to conduct R&D activities that expand the variety of goods. We assume that the monopolistically competitive firms, which are subject to borrowing constraints, produce intermediate goods and conduct R&D activities.

4.1 Basic setup

A representative household owns a mass of firms, indexed by $i \in [0, N_{t-1}]$, that produce intermediate goods, where N_{t-1} measures the varieties of intermediate goods in period t. Firm i produces variety i monopolistically and can borrow funds from the household. In what follows, we omit the bank, for simplicity. The final good is produced competitively from intermediate goods $y_{i,t}$ by the following production function:

$$Y_t = \left(\int_0^{N_{t-1}} y_{i,t}^{\eta} di \right)^{\frac{1}{\eta}},$$

where $0 < \eta < 1$. Because the final good producer maximizes $Y_t - \int_0^{N_{t-1}} p_{i,t} y_{i,t} di$, where $p_{i,t}$ is the real price of intermediate good i, perfect competition in the final goods market implies that

$$p_{i,t} = p(y_{i,t}) = A_t y_{i,t}^{\eta-1},$$

where

$$A_t \equiv Y_t^{1-\eta}.$$

Firm i produces intermediate good i from capital $k_{i,t}$ and labor $l_{i,p,t}$ by the following production function:

$$y_{i,t} = k_{i,t}^{\alpha} l_{i,p,t}^{1-\alpha}.$$

Each firm i employs labor $l_{i,t}$ and capital $k_{i,t}$, produces intermediate goods $y_{i,t}$ from $l_{i,p,t}$ $(\leq l_{i,t})$ and $k_{i,t}$, and conducts R&D with labor input $l_{i,t} - l_{i,p,t}$. The R&D activity creates

$$\kappa \bar{N}_{t-1}(l_{i,t} - l_{i,p,t})$$

units of new varieties of intermediate goods, where κ is the parameter that represents the efficiency of R&D activity and $\bar{N}_{t-1}$ is the social level of the variety, which represents the externality from the stock of knowledge on the R&D activity. This externality ensures the existence of the balanced growth path (BGP). The law of motion for the measure of varieties is written as follows:

$$N_t = N_{t-1} + \kappa \bar{N}_{t-1}(L_t - L_{p,t}),$$

where $L_t = \int_0^{N_{t-1}} l_{i,t} di$ and $L_{p,t} = \int_0^{N_{t-1}} l_{i,p,t} di$. When a new variety is created, a new monopolistic firm that produces the variety is also born. Each new variety is produced by a newborn firm. The parent firm creates newborn firms and then treats them as members of its own dynasty. As the parent and newborn firms are technologically identical, the burden of inter-period debt for the parent firm is shared equally by all firms in the dynasty. In this general equilibrium model, the state of nature x_t is given by $x_t = (N_{t-1},\ \{b_{i,t-1}\}_{i=0}^{N_{t-1}})$, because prices are the equilibrium outcomes. Exogenous redistribution shocks may hit the distribution of debt, $\{b_{i,t-1}\}_{i=0}^{N_{t-1}}$, and make x_t evolve stochastically. The transition of states is a Markov process, which is determined endogenously in equilibrium, and is taken as given by the individual households and firms. To simplify the notation, we use time subscript t instead of state x_t and simply omit x_t in what follows, on the understanding that all variables and functions with subscript t depend on state x_t. Thus, the value of the firm is determined by the following dynamic programming, where we omit subscript i, for simplicity:

$$V_t(b_{t-1}) = \max \pi_t + \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \left\{ 1 + \kappa \bar{N}_{t-1}(l_t - l_{p,t}) \right\} V_{t+1}(b_t) \right],$$

subject to

$$\pi_t = A_t k_t^{\alpha\eta} l_{p,t}^{(1-\alpha)\eta} - w_t l_t - r_t^K k_t - b_{t-1} + [1 + \kappa \bar{N}_{t-1}(l_t - l_{p,t})] \frac{b_t}{R_t}, \tag{23}$$

$$w_t l_t + r_t^K k_t \leq \phi A_t k_t^{\alpha\eta} l_{p,t}^{(1-\alpha)\eta} + \max \left\{ \xi S_t - \frac{b_t}{R_t},\ 0 \right\} [1 + \kappa \bar{N}_{t-1}(l_t - l_{p,t})], \tag{24}$$

$$\pi_t \geq 0, \tag{25}$$

$$l_t \geq l_{p,t}, \tag{26}$$

$$b_t \leq b_{z,t},$$

where S_t and $b_{z,t}$ are taken as given, and are the equilibrium outcomes, as we see shortly. The Lagrange multipliers, $\lambda_t,\ \mu_t,\ \lambda_{\pi,t}$, and $\lambda_{l,t}$, are associated with the budget constraint (23), borrowing constraint (24), non-negativity of profits (25), and non-negativity of labor input for R&D (26), respectively. In equilibrium, the liquidation value S_t and the upper limit of borrowing $b_{z,t}$ are given in the same manner as in the previous section. Thus, the equilibrium condition that determines S_t is

$$S_t = \max_b \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} V_{t+1}(b) \right] + \frac{b}{R_t}.$$

The value of $b_{z,t} = b_z(x_t)$ is given by

$$b_z(x_{t-1}) = \inf_{x_t \in \Lambda(x_{t-1})} (1-\phi) f_t(\sigma_z(x_t), x_t) - \varphi(0) + \frac{b_z(x_t)}{R_t},$$

where $f(\sigma, x)$ and $\sigma_z(x)$ are the same as those in Section 3, and we assume the parameter restriction (9).

A representative household solves the following problem:

$$\max_{C_t, L_t, B_t, K_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \right],$$

subject to the budget constraint

$$C_t + K_t - \frac{B_t}{1+r_t} \le w_t L_t + (r_t^K + 1 - \rho) K_{t-1} - B_{t-1} + T_t,$$

where β is the subjective discount factor, C_t is consumption, L_t is total labor supply, K_t is capital stock, ρ is the depreciation rate of capital, B_t is inter-period lending to the firms, and T_t is a lump-sum transfer, which consists of the tax and dividends. The period utility is

$$U(C, L) = \frac{\left[C^{\frac{1}{1+\gamma}} (1-L)^{\frac{\gamma}{1+\gamma}} \right]^{1-\theta}}{1-\theta}.$$

Let m_t be the Lagrange multiplier associated with the budget constraint for the representative household, which is given by the FOC with respect to C_t:

$$m_t = \beta^t \frac{\partial}{\partial C} U(C_t, L_t).$$

The FOC with respect to K_t and B_t implies

$$\frac{1}{1+r_t} = \mathbb{E}_t \left[\frac{1}{r_{t+1}^K + 1 - \rho} \right] = \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \right].$$

The market-clearing conditions are

$$\begin{aligned}
&C_t + K_t - (1-\rho) K_{t-1} = Y_t, \\
&\int_0^{N_{t-1}} l_{i,t} di = L_t, \\
&\int_0^{N_{t-1}} l_{i,p,t} di = L_{p,t}, \\
&\int_0^{N_{t-1}} k_{i,t} di = K_{t-1}, \\
&\int_0^{N_{t-1}} \frac{b_{i,t}}{R_t} di = \frac{B_t}{1+r_t}, \\
&N_t = N_{t-1} + \kappa \bar{N}_{t-1} (L_t - L_{p,t}).
\end{aligned}$$

The equilibrium condition is

$$\bar{N}_t = N_t.$$

Competitive equilibrium.— A competitive equilibrium consists of sequences of prices $\{r_t, r_t^K, w_t, m_t\}$, a household's decisions $\{C_t, L_t, K_t\}$, firms' decisions $\{\pi_t, l_t, l_{p,t}, k_t, b_t\}$, and a measure of varieties N_t, such that *(i)* the representative household and firms solve their respective optimization problems, taking prices and N_{t-1} as given; and *(ii)* the market-clearing conditions and equilibrium condition are all satisfied.

Debt-ridden firms: A firm that owes the maximum amount $b_{z,t}$ is called a *debt-ridden firm* in a similar manner to the previous section.

4.2 BGP without debt-ridden firms

In what follows, we assume for simplicity that $\theta \to 1$ and

$$U(C, L) = \ln C + \gamma \ln(1 - L).$$

In this subsection, we focus on the deterministic case where state x_t evolves deterministically, and then characterize the BGP on which all firms are *normal.* The firms are normal when the dividend is positive: $\pi_t > 0$ for all t. On the BGP, labor and the growth rate are constant: $L_t = L$ and $N_t/N_{t-1} = g$. We define

$$E \equiv \frac{1-\eta}{(1-\alpha)\eta}.$$

We guess that $Y_t = Y \times N_{t-1}^E$, $A_t = A \times N_{t-1}^{(1-\eta)E}$, $C_t = C \times N_{t-1}^E$, $K_t = K \times N_t^E$, $w_t = w \times N_{t-1}^E$, $l_t = L/N_{t-1}$, $k_t = k \times N_{t-1}^{E-1}$, $V_t = V \times N_{t-1}^{E-1}$, and $\pi_t = \pi \times N_{t-1}^{E-1}$. The FOCs and constraints imply that there exists a unique BGP, which is given in Appendix D. For the numerical simulation, we set the parameters to the values for Japan, the United States, and the European Union, shown in Table 2. Note that we set $\beta = 0.98$ because it is an annual model.[14] We adopt a simulation strategy under which the parameters γ and κ are calculated endogenously, such that the two variables L and g_{TFP} attain the target values. Thus, the economic growth rate on the BGP is 2.51% per year for the Japanese economy.[15] It is easily confirmed numerically that there exists a BGP for a wide range of parameter settings.

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[14] Hayashi and Prescott (2002) estimated $\beta = 0.976$ and Sugo and Ueda (2008) used $\beta = 0.98$ for the annual discount rate of the Japanese economy.

[15] The economic growth rate on the BGP is given by $g^E = g_{TFP}^{\frac{1-\eta}{(1-\alpha)\eta}\frac{\eta}{1-\eta}} = g_{TFP}^{\frac{1}{1-\alpha}} = 1.0251$, where g is the growth rate of N_t and $E = \frac{1-\eta}{(1-\alpha)\eta}$.

Common parameters

Parameter	Economic interpretation	value
β	the subjective discount factor	0.98
ρ	the depreciation rate of capital	0.1
η	the parameter for the aggregation function	0.7

Country-specific parameters

Parameter	Economic interpretation	Japan	US	EU
α	the share of labor in production	0.31	0.34	0.37
γ	the inverse of the elasticity of labor supply	1.50	1.80	1.97
κ	the efficiency of R&D	0.39	0.37	0.34
ϕ	the collateral ratio of revenue	0.50	0.50	0.47
τ	the tax advantage for debt	0.30	0.35	0.35
ξ	the collateral ratio of the foreclosure value	0.08	0.27	0.09
L	the total labor supply in the steady state	0.36	0.31	0.28
g_{TFP}	the growth rate of TFP in the steady state	1.017	1.015	1.012
z	the ratio of debt-ridden firms at the shock period	0.940	0.607	0.838

Table 2: Parameter settings

4.3 Low growth equilibrium with debt-ridden firms

Now, we consider the equilibrium where some firms are debt-ridden and the others are normal. In this subsection, we again focus on the deterministic case, where state x_t evolves deterministically. We assume that firms $i \in [0, Z_t]$ are debt-ridden and firms $i \in (Z_t, N_t]$ are normal. Define $z_t \equiv \frac{Z_t}{N_t}$. The initial value of z_t, denoted by z, is given exogenously. In our simulation, we assume $z < 1$ and $\frac{b_{z,t}}{R_t} > \xi S_t$ for debt-ridden firms.[16]

Numerical experiment: We can calculate the equilibrium dynamics numerically using a full nonlinear method. Linearization is not necessary for the deterministic simulation (see Appendix E for detrending and Appendix F for the details of the calculation of the dynamics). Figure 5 shows the results of the numerical simulation in which the economy is initially on the BGP, where $Z_t = 0$, and an unexpected redistribution shock hits the economy in period 10, making $z_{10} = z = 0.94$, where z_t is defined by $z_t \equiv Z_t/N_t$. In other words, the sudden redistribution of wealth from firms to households makes 94.0% of

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[16] We cannot find the parameter set that makes $z = 1$ and $\frac{b_{z,t}}{R_t} > \xi S_t$ for debt-ridden firms simultaneously. If both $z = 1$ and $\frac{b_{z,t}}{R_t} > \xi S_t$ hold simultaneously, there would exist a zero growth path on which all firms are debt-ridden and the economic growth rate is zero, because no firms conduct R&D activities. In a simplified model, in which capital k does not exist and labor is the only input, it is proven analytically that $z = 1$ and $\frac{b_{z,t}}{R_t} > \xi S_t$ cannot hold simultaneously for any parameter set.

all firms debt-ridden in period 10. The parameter values for Figure 5 are given as those for Japan in Table 2. The three sets of parameter values are calibrated to fit the data of Japan, the United States, and the European Union (EU), respectively, according to the method described in Appendix G, where we identify and assess the parameter region that we use in our numerical experiments. The features of the equilibrium path shown in Figure 5 are as follows:

- Slowdown of economic growth: Borrowing constraints are tighter after the buildup of debt. Thus, the aggregate inputs decrease and economic growth slows for an extended period.
- Decrease in net entry: The growth rate of the number of firms, $g_t = N_t/N_{t-1}$, decreases. This feature is consistent with the observation that TFP and the net entry of firms decreased in Japan in the 1990s.
- Buildup of NPLs: In this example, there are Z_t debt-ridden firms and their debt stays at an inefficiently high level. This feature is consistent with the historical episodes of persistent stagnation with overly indebted firms and/or households, such as Japan in the 1990s.
- Labor wedge reduction: In this example, the labor wedge, $1-\tau_L$, diminishes persistently as a direct consequence of the tightening of the aggregate borrowing constraint on working capital loans for wage payments. This tighter borrowing constraint creates a larger gap between the wage rate and marginal product of labor. The gap is measured by τ_L. In this way, the persistent reduction in the labor wedge observed in the aftermath of a financial crisis can be accounted for by the emergence of debt-ridden firms.[17]

Next, we calibrate and conduct numerical simulations for the United States and the EU.[18] The parameter values are given in Table 2. Figure 6 compares the TFP in the

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[17] As Chari et al. (2007) posited, the labor wedge, $1-\tau_{L,t}$, is defined by $1-\tau_{L,t} = \frac{MRS_t}{MPL_t}$, where $MRS_t = \frac{\gamma C_t}{1-L_t} = w_t$ and $MPL_t = \frac{\alpha Y_t}{L_t}$ in our model. Thus, the labor wedge can be calculated by $1-\tau_{L,t} = \frac{w_t L_t}{\alpha Y_t}$. In our model, the labor wedge $1-\tau_L$ is proportional to the labor share. Thus, both an economic slowdown and a shrinkage of the labor share (from the buildup of debt) are observed simultaneously in our model. This feature of our model contrasts with the countercyclicality of the labor share in business cycle frequencies (Schneider, 2011). However, our model seems compatible with countercyclicality in the short run. In our model, the buildup of debt causes the long-term variations in the labor wedge, whereas short-run countercyclicality can be caused by factors such as productivity shocks in business cycle frequencies (Ríos-Rull and Santaeulàlia-Llopis, 2010).

[18] The EU comprises the following 28 countries: Austria, Belgium, Bulgaria, Croatia, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, the Netherlands, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden, and the United Kingdom.

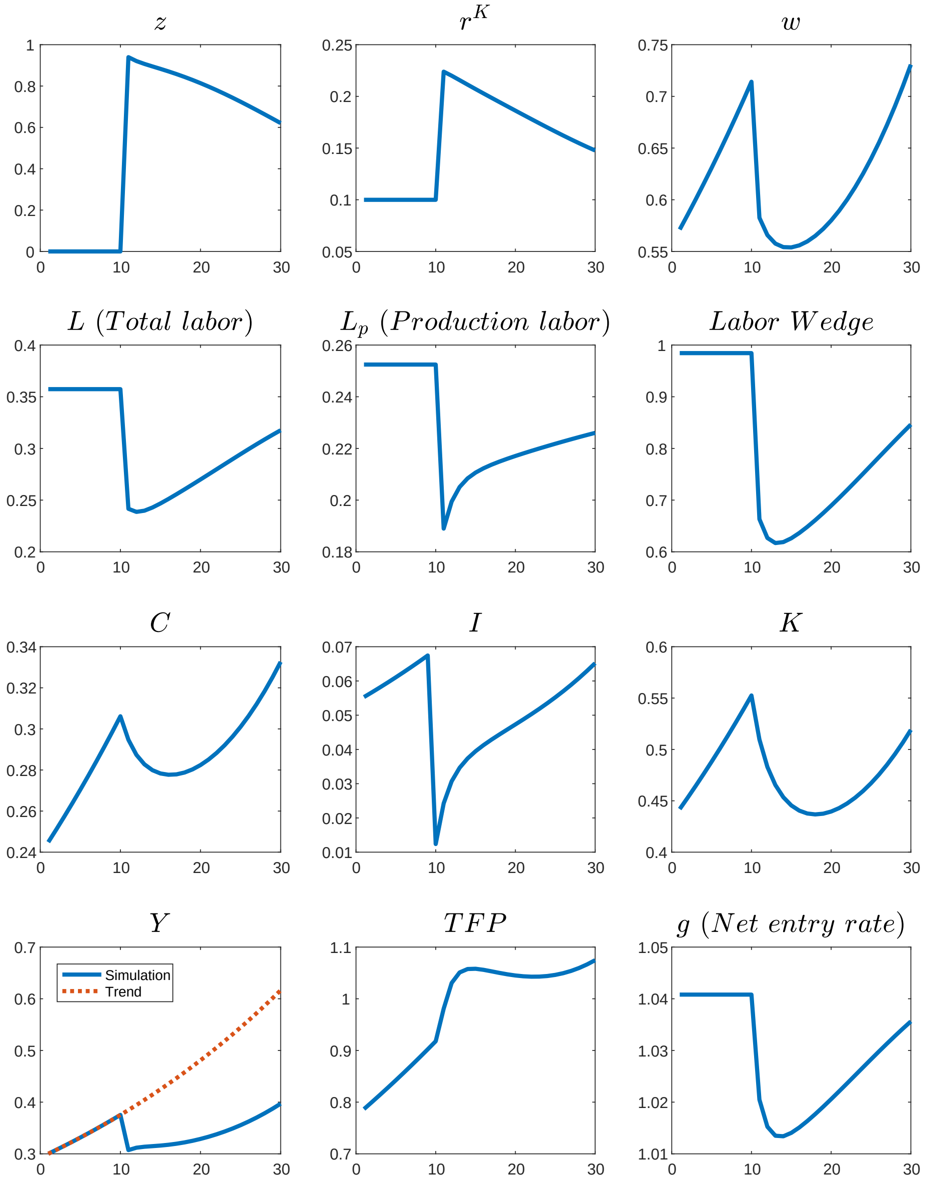

Figure 5: Responses to a buildup of debt (Japan)

model with the actual TFP in Japan, the United States, and the EU, where the TFP in the model is defined by

$$TFP_t = \frac{Y_t}{K_{t-1}^{\alpha} L_t^{1-\alpha}}.$$

Similarly, Figure 7 compares the output of the numerical experiment with actual per capita GDP. We assume that the unexpected shock hits the economy in period 10 of the simulation, which corresponds to the asset-bubble collapse in 1990, in the case of Japan, and the financial crisis in 2009, in the case of the United States and EU. For the United States and EU, we extend the observed variables to 2025. We posit that the variables grow in future periods by constant growth rates, which are equal to the average growth rates in 2011–2015 for the United States and 2011–2014 for the EU. These extensions are based on the implicit assumption that the US and EU economies have fallen into decade-long stagnation.[19] We compare our simulation results with the extended data on the United States and EU because a goal of our numerical experiment is to examine the capability of our model to account for decade-long recessions in the aftermath of financial crises. The figures show that the model fits the growth rate data fairly well, except for the impact period, when we observe large spikes because of a large debt shock. Although the spikes are large in our frictionless economy, they could be much smaller if realistic frictions in markets and nominal rigidities are incorporated into the model. In any case, the numerical simulation of our model shows the overall slowdown of productivity and economic growth resulting from the debt shock, indicating the usefulness of our model in accounting for persistent recessions in the aftermath of financial crises.

4.4 Policy implications

The policy implications of the results presented in this paper seem noteworthy from a practical point of view. The shocks that cause persistent stagnation are exogenous technological changes in the existing literature. In our model, the one-time buildup of debt tightens the borrowing constraint and causes a persistent slowdown in economic growth, while there is no technological change. Thus, our model implies that reducing overly accumulated debt can restore economic growth. Note that the physical liquidation of debt-ridden borrowers is not necessary, but their relief from excessive debt restores their efficiency and high economic growth at the aggregate level. This policy implication contrasts sharply with the findings of prior studies, in which debt reduction, per se, has little effect, and policymakers can only mitigate recessions by implementing accommodative monetary and fiscal policies or designing ex-ante financial regulations.

Our model implies that it is optimal for lenders to keep borrowers debt-ridden if the

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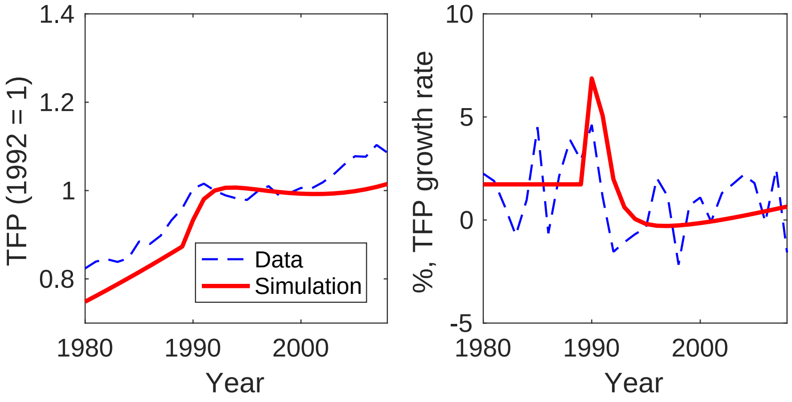
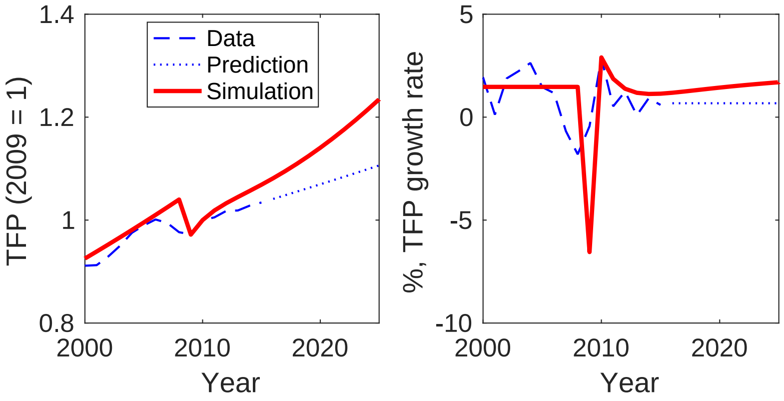
[19] We implicitly assume that the US and EU economies have fallen into the “great depressions” described in Kehoe and Prescott’s (2007) book.

Japan



United States



EU

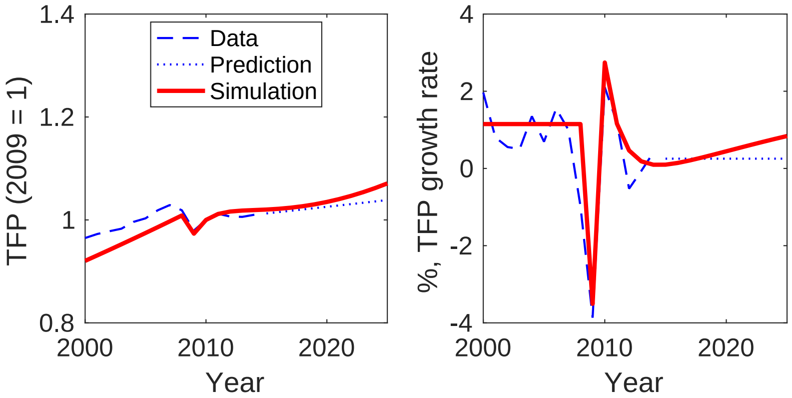


Figure 6: TFP for Japan, the United States, and the EU: Comparison between the data and the simulation

Note: In Japan, TFP is classified as the "market economy" sectors, which excludes education, medical services, government activities, and imputed house rent.
Sources: Our calculation; The Research Institute of Economy, Trade and Industry, *JIP 2014 database*; Fernald (2012); European Commission, *AMECO*

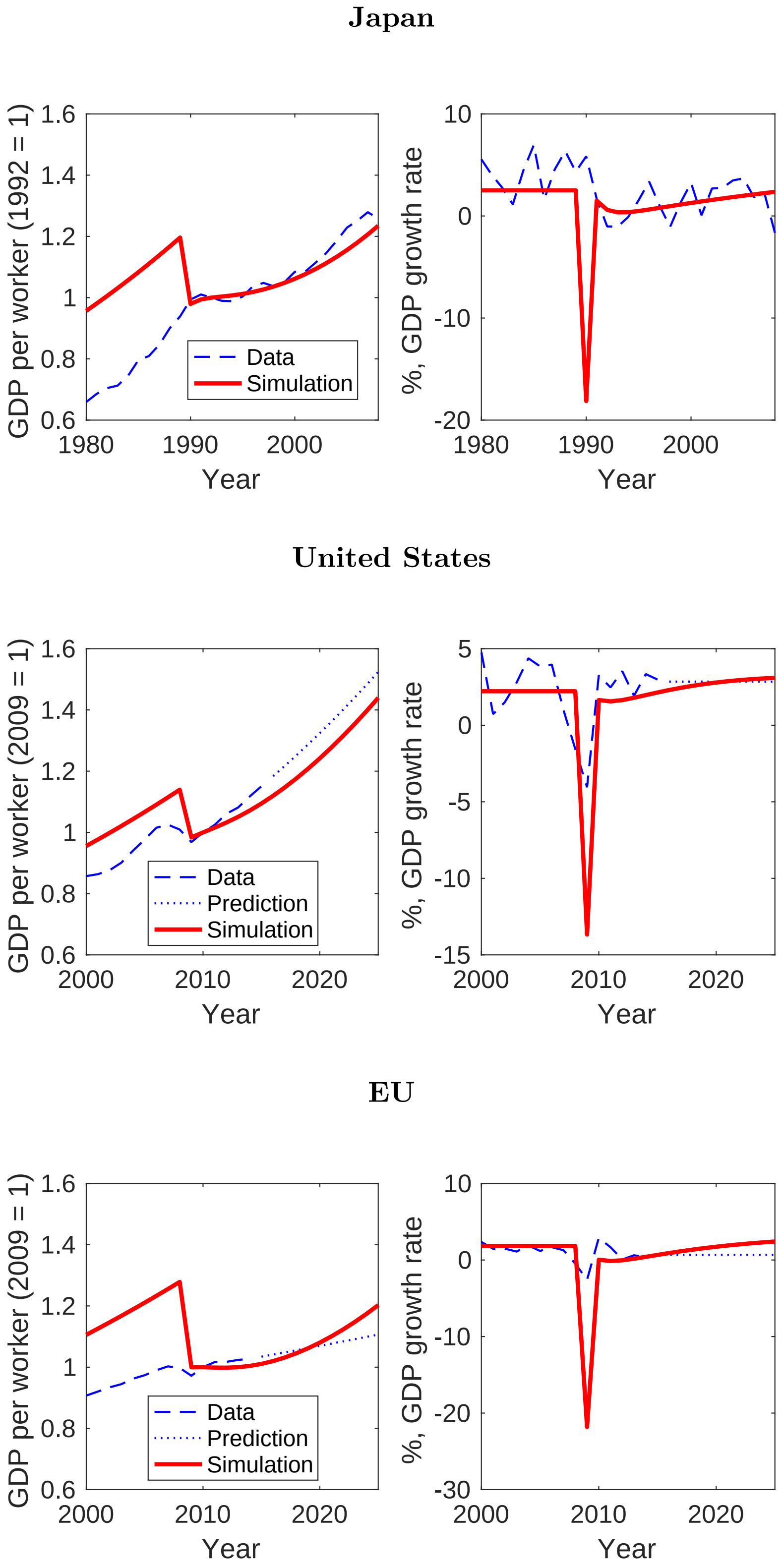


Figure 7: GDP for Japan, the United States, and the EU: Comparison between the data and the simulation

Sources: Our calculation; Cabinet Office, Government of Japan, *Annual Report on National Accounts*; Fernald (2012); European Commission, *AMECO* outstanding debt is large. Lenders have no incentive to reduce their loans to debt-ridden borrowers, while their inaction protracts the aggregate inefficiency. Thus, policy interventions by the government can be effective in restoring economic growth by promoting debt restructuring or wealth redistribution from lenders to borrowers. Policy measures may include regulatory reforms to make bankruptcy procedures less costly and debtor friendly, and to promote debt-for-equity swaps to reduce outstanding debt, as well as the injection of funds as a subsidy or equity to banks that forgive debt and write off NPLs. The injection of a bank subsidy or equity is usually interpreted as bank recapitalization, because the banks become insolvent in most cases when a substantial number of their borrowers fall into a debt-ridden state. This policy implication is straightforward and robust in our model, and seems reasonable from our experience of Japan's lost decade of the 1990s, the Great Recession in the United States, and subsequent debt crises in Europe, whereas existing models may not clearly imply that borrowers' relief from excessively accumulated debt is good for a crisis-hit economy.

5 Conclusion

Decade-long recessions are often observed after financial crises. In particular, the "secular stagnation" hypothesis has drawn much attention recently. In this study, we hypothesized that the buildup of large corporate debt causes a persistent economic slowdown. Economic agents can become overly indebted because of, for example, the boom and bust of asset-price bubbles. By analyzing the economy with endogenous borrowing constraints, where lenders can forgive defaulting borrowers, we showed that borrowers with the maximum repayable debt fall into a debt-ridden state. In this state, they can repay only the interest and cannot reduce the principal of the debt, which means they continue inefficient production forever.

The emergence of a substantial number of debt-ridden borrowers lowers economic growth by tightening the aggregate borrowing constraint. This tightening of aggregate borrowing constraints owing to the mass emergence of debt-ridden borrowers may manifest as a "financial shock," during or after a financial crisis. Because lenders have no incentive to reduce their loans to debt-ridden borrowers, a government intervention to facilitate debt restructuring (i.e., relief for debt-ridden borrowers from their excessive debt) may be necessary to enhance economic growth when the economy falls into persistent stagnation in the aftermath of a financial crisis. This policy implication is in a stark contrast to those of existing literature in which debt reduction, per se, has little effect, and policymakers can only mitigate recessions by conducting accommodative monetary and fiscal policies or designing the ex-ante financial regulations.

The endogenous borrowing constraint of this study has a unique feature that debt tightens the constraint and generates inefficiency, which can be permanent in the deterministic case. Therefore, it may serve as a useful building block for business cycle models, thereby enriching aggregate dynamics. Broader applications are left for future research.

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Appendices

A Proof of Lemma 2

Proof of (i) is as follows. As $\tau = 0$, the envelope condition (17) implies that $\frac{\partial}{\partial b} W(b, x) = -\lambda_\pi \leq 0$, that is, $W(b_{-1}, x)$ is weakly decreasing in b_{-1}. (ii) follows immediately from the fact that $B^*(x; S) - b_{-1}$ is the dividend when $b_{-1} < B^*(x; S)$ and the firm chooses $\sigma = \sigma^*(x)$, and that Assumption 2 implies that if $\sigma_t = \sigma^*(x_t)$, then $\sigma_{t+j} = \sigma^*(x_{t+j})$, for all j. (iii) is shown as follows. When $B^*(x; S) < b_{-1} \leq B_z(x; S)$, we have $\frac{b^*(x;S)}{R(x)} < \frac{b(x;S)}{R(x)} \leq \xi S(x)$. This inequality and the definition of $b^*(x; S)$ imply that the borrowing constraint is tight and $\sigma(x) < \sigma^*(x)$. Thus, $W(b_{-1}, x) < W^*(x)$. Now, suppose that $\pi > 0$ in this case. Then, the firm can relax the borrowing constraint by reducing π and b, and increase $W(b_{-1}, x)$. This is a contradiction. Therefore, π should be zero. When $b_{-1} > B_z(x; S)$, it is obvious that $W(b_{-1}, x) < W^*(x)$, and there exists T, such that the borrowing constraint is $\sigma_t \leq \phi f(\sigma_t, x_t)$ for $0 \leq t < T$, and $\sigma_t \leq \phi f(\sigma_t, x_t) + \xi S_t - \frac{b_t}{R_t}$ for $t \geq T$, in the sequential problem corresponding to the Bellman equation (11). Suppose $\pi_0 > 0$. Then, the firm can relax the borrowing constraint at T by reducing π_0 and b_0, and increase $W(b_{-1}, x)$. This is a contradiction. Thus, π_0 should be zero.

B Proof of Proposition 5

We prove Proposition 5 using the Schauder fixed point theorem (see, for example, Theorem 17.4 in Stokey and Lucas with Prescott, 1989). First, we define several parameters.

Definition 2. Define $r_{\min}$, $R_{\max}$, and $f_{\max}$ as follows:

$$\begin{aligned} r_{\min} &= \inf_{x \in \Lambda} r(x), \\ R_{\max} &= \sup_{x \in \Lambda} R(x), \\ f_{\max} &= \sup_{x \in \Lambda} \frac{\partial}{\partial \sigma} f(\sigma_z(x), x). \end{aligned}$$

Note that $f_{\max}$ is the upper bound for $\frac{\partial}{\partial \sigma} f(\sigma, x)$ because $\sigma \geq \sigma_z(x)$, for all x, and $f(\sigma, x)$ is increasing and concave in σ. It is easily shown that

$$f_{\max} = \frac{\eta}{\phi}.$$

Proposition 1 shows that, given $S(x_t)$, there exists a solution to (6) in which the value function can be denoted as $V(b, x; S)$. Define operator T that maps $S(x_t)$ to $TS(x_t)$, where

$$TS(x_t) \equiv \max_b \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} V(b, x_{t+1}; S) \right] + \frac{b}{R_t}.$$

If mapping $T : G \to G$ is continuous, and family $T(G)$ is equicontinuous, then the Schauder fixed point theorem applies, and it is shown that T has a fixed point in G. Then, T is obviously a mapping from G to G, because $T(G) \subseteq G$. The continuity of T follows directly from the continuity of $V(b, x; S)$ with respect to S, which is shown in the following lemma.

Lemma 10. *$V(b, x; S)$ is continuous with respect to S, that is, for all $\varepsilon > 0$, there exists $\delta > 0$, such that if $|S - S'| < \delta$ for any S, $S' \in G$, then $|V(b, x; S) - V(b, x; S')| < \varepsilon$, where the norms are the sup norm.*

Proof. The equivalence between the functional equation (6) and the corresponding infinite horizon problem shown in Theorem 9.2 of Stokey and Lucas with Prescott (1989) implies that

$$
\begin{aligned}
V(b,x;S) = \max_{\{\sigma_t,b_t\}_{t=0}^{\infty}} \mathbb{E}_0 &\left[\sum_{t=0}^{\infty} \frac{m_t}{m_0}\pi_t\right], \\
\text{subject to} \qquad & \pi_t = f(\sigma_t,x_t) - \sigma_t - b_{t-1} + \frac{b_t}{R_t}, \\
& \sigma_t \le \phi f(\sigma_t,x_t) + \max\left\{\xi S(x_t) - \frac{b_t}{R_t},\ 0\right\}, \\
& b_t \le b_z(x_t), \\
& \pi_t \ge 0, \\
& b_{-1} = b, \\
& x_0 = x.
\end{aligned}
$$

The solution to the above problem is a path: $\{\sigma_t, b_t\}_{t=0}^{\infty}$. Assumptions 1 and 2 directly imply that there are three stages in this path, that is, given b_{-1}, there exist two integers, t_1 and t_2, where $0 \le t_1 \le t_2 \le \infty$, such that

- Stage 1 ($B_{z,t} < b_{t-1} \le \bar{B}_{z,t}$): $\xi S(x_t) < \frac{b_t}{R_t}$ for $0 \le t < t_1$;
- Stage 2 ($B_t^{ce} < b_{t-1} \le B_{z,t}$): $\xi S(x_t) \ge \frac{b_t}{R_t}$ and $\sigma_t < \sigma^{ce}(x_t)$ for $t_1 \le t < t_2$;
- Stage 3 ($b_{t-1} \le B_t^{ce}$): $\xi S(x_t) \ge \frac{b_t}{R_t}$ and $\sigma_t = \sigma^{ce}(x_t)$ for $t \ge t_2$.

Note that the values of t_1 and t_2 are history-dependent, but the history dependence does not affect the following proof. Assumption 1 implies that once the economy enters Stage 2, it never returns to Stage 1. Assumption 2 implies that once the economy enters Stage 3, it never returns to Stage 2. As we see in Section 3.3, $\pi_t = 0$ in Stages 1 and 2, and $\pi_t > 0$ in Stage 3.

Now, we evaluate $|V(b,x;S) - V(b,x;S+\delta/\xi)|$, where $S + \delta/\xi = S(x) + \delta/\xi$ for a small $\delta > 0$. We use $\{\sigma_t(\delta), b_t(\delta)\}_{t=0}^{\infty} = \{\sigma_t + \hat{\sigma}_t(\delta), b_t - \hat{b}_t(\delta)\}_{t=0}^{\infty}$ to denote the solution to the above infinite horizon problem, with S replaced by $S + \delta/\xi$. The corresponding dividend stream is written as $\{\pi_t(\delta)\}_{t=0}^{\infty} = \{\pi_t + \hat{\pi}_t(\delta)\}_{t=0}^{\infty}$. The boundary of stages, $\{t_1, t_2\}$, changes to $\{t_1(\delta), t_2(\delta)\}$. We assume that δ is sufficiently small, such that

$$
(\Theta + 1)\delta < \inf_{x\in\Lambda} (1-\phi) f(\sigma_z(x), x) - \nabla, \tag{27}
$$

where Θ is defined in (34) below. Note that Assumption 3 warrants the existence of $\delta\,(> 0)$.

Stage 1: For $t \leq t_1(\delta)$, $\frac{b_t}{R_t} > \xi S_t + \delta$. Thus, for $t \leq t_1(\delta)$, the change from ξS to $\xi S + \delta$ does not change the variables: $(\sigma_t(\delta), b_t(\delta)) = (\sigma_t, b_t)$ or $(\hat{\sigma}_t(\delta), \hat{b}_t(\delta)) = (0, 0)$. The dividend is also zero: $\pi_t(\delta) = \pi_t = 0$. Therefore, as (27) implies that $\delta < \inf_{x \in \Lambda} f(\sigma_z(x), x) - \sigma_z(x)$, it is obvious that $t_1 - 1 \leq t_1(\delta) \leq t_1$. We evaluate the value of $b_{t_1-1}(\delta)$, which is the initial debt at the beginning of t_1, in the case where ξS_t changes to $\xi S_t + \delta$ in the above borrowing constraint.

- **Case where $t_1(\delta) = t_1$:**
  In this case, it is obvious that $b_t(\delta) = b_t$ for all $t \leq t_1 - 1$. Thus, $b_{t_1-1}(\delta) = b_{t_1-1}$.

- **Case where $t_1(\delta) = t_1 - 1$:**
  In this case, $b_{t_1-2}(\delta) = b_{t_1-2}$ and $\xi S_{t_1-1} \geq \frac{b_{t_1-1}(\delta)}{R_{t_1-1}}$. The budget and borrowing constraints in period $t_1 - 1$ imply

$$b_{t_1-2} - \xi S_{t_1-1} - \delta = (1+\phi) f(\sigma_{t_1-1}(\delta), x_{t_1-1}) - 2\sigma_{t_1-1}(\delta). \tag{28}$$

In the original problem with ξS_t, the budget and borrowing constraints imply

$$b_{t_1-2} - \xi S_{t_1-1} > (1+\phi) f(\sigma_{z,t_1-1}, x_{t_1-1}) - 2\sigma_{z,t_1-1}. \tag{29}$$

The parameter restriction that $\frac{\eta}{2-\eta} < \phi$ in Assumption 3 implies that $(1+\phi)\frac{\partial}{\partial \sigma} f(\sigma, x) - 2 \leq (1+\phi) f_{\max} - 2 < 0$, which implies that function $(1+\phi) f(\sigma, x) - 2\sigma$ is decreasing in σ. Thus, $\hat{\sigma}_{t_1-1}(\delta) > 0$, and (28) and (29) imply

$$0 \leq (1+\phi) f(\sigma_{z,t_1-1}, x_{t_1-1}) - 2\sigma_{z,t_1-1} - [(1+\phi) f(\sigma_{t_1-1}(\delta), x_{t_1-1}) - 2\sigma_{t_1-1}(\delta)] \leq \delta. \tag{30}$$

The concavity of $f(\sigma, x)$ and $(1+\phi)\frac{\partial}{\partial \sigma} f(\sigma, x) - 2 < 0$ imply that

$$\begin{aligned} &(1+\phi) f(\sigma_{t_1-1}(\delta), x_{t_1-1}) - 2\sigma_{t_1-1}(\delta) < \\ &\qquad (1+\phi) f(\sigma_{z,t_1-1}, x_{t_1-1}) - 2\sigma_{z,t_1-1} - [2 - (1+\phi) f_{\max}]\hat{\sigma}_{t_1-1}(\delta). \end{aligned} \tag{31}$$

Inequalities (30) and (31) imply that

$$0 \leq \hat{\sigma}_{t_1-1}(\delta) \leq \frac{\delta}{2 - (1+\phi) f_{\max}}.$$

Then, $\frac{b_{t_1-1}(\delta)}{R_{t_1-1}} = \phi f(\sigma_{t_1-1}(\delta), x_{t_1-1}) - \sigma_{t_1-1}(\delta) + \xi S_{t_1-1} + \delta$ must satisfy

$$\left[\frac{1 - (1+\phi) f_{\max}}{2 - (1+\phi) f_{\max}}\right] \delta \leq \frac{b_{t_1-1}(\delta)}{R_{t_1-1}} - \frac{b_{t_1-1}}{R_{t_1-1}} \leq \left[\frac{2 - f_{\max}}{2 - (1+\phi) f_{\max}}\right] \delta.$$

Therefore, there exists a positive number C_{-1} such that $b_{t_1-1} - C_{-1}\delta < b_{t_1-1}(\delta) < b_{t_1-1} + C_{-1}\delta$.

Stage 2: Note that $\pi_t(\delta) = 0$ for $t \leq t_2(\delta) - 1$. We denote $t = t_1 + n$ and count time by n. The above analysis of Stage 1 implies that $b_{t_1-1}(\delta)$ satisfies that $b_{t_1-1} - C_{-1}\delta < b_{t_1-1}(\delta) < b_{t_1-1} + C_{-1}\delta$. We change the subscript for the variables from t to n. Define $\bar{n}$ as the solution to

$$\xi\bar{\omega} = \sum_{i=1}^{\bar{n}} \frac{(1-\phi)\underline{f}_z}{(1+r_{\max})^i}.$$

Note that Assumption 1 warrants that this equation has a finite solution $\bar{n}$, and that $\bar{n}$ does not depend on $S(x)$. Then,

$$t_2(\delta) - t_1(\delta) \leq \bar{n}.$$

In period $n = 0$, which corresponds to $t = t_1$, we have $f(\sigma_0(\delta)) - \sigma_0(\delta) \leq f(\sigma_0) - \sigma_0 + (f_{\max} - 1)\hat{\sigma}_0(\delta)$. The budget constraint implies that $\frac{\hat{b}_0(\delta)}{R_0} \leq (f_{\max} - 1)\hat{\sigma}_0(\delta) + C_{-1}\delta$. Then, the borrowing constraint implies that $\hat{\sigma}_0(\delta) \leq \phi f_{\max}\hat{\sigma}_0(\delta) + \delta + (f_{\max} - 1)\hat{\sigma}_0(\delta) + C_{-1}\delta$, which can be rewritten as $\hat{\sigma}_0(\delta) = \frac{(1+C_{-1})\delta}{2-(1+\phi)f_{\max}}$. We define

$$A_0 \equiv \frac{f_{\max} - 1}{2 - (1+\phi)f_{\max}} \ (> 0).$$

Then, $(f_{\max} - 1)\hat{\sigma}_0(\delta) = (1 + C_{-1})A_0\delta$ and $\frac{b_0(\delta)}{R_0} \geq \frac{b_0}{R_0} - (1 + C_{-1})A_0\delta$. In period $n = 1$, $f(\sigma_1(\delta)) - \sigma_1(\delta) \leq f(\sigma_1) - \sigma_1 + (f_{\max} - 1)\hat{\sigma}_1(\delta)$. The budget constraint implies that $\frac{\hat{b}_1(\delta)}{R_1} \leq (f_{\max} - 1)\hat{\sigma}_1(\delta) + (1 + C_{-1})R_{\max}A_0\delta$. Then, the borrowing constraint implies that $\hat{\sigma}_1(\delta) \leq \phi f_{\max}\hat{\sigma}_1(\delta) + \delta + (f_{\max} - 1)\hat{\sigma}_1(\delta) + (1 + C_{-1})R_{\max}A_0\delta$, which can be rewritten as $\hat{\sigma}_1(\delta) = \frac{[1+(1+C_{-1})R_{\max}A_0]}{2-(1+\phi)f_{\max}}\delta$. We define

$$A_1 \equiv A_0[1 + (1 + C_{-1})R_{\max}A_0].$$

Then, $(f_{\max} - 1)\hat{\sigma}_1(\delta) = A_1\delta$ and $\frac{b_1(\delta)}{R_1} \geq \frac{b_1}{R_1} - [(1 + C_{-1})R_{\max}A_0 + A_1]\delta$. In period $n = 2$, $f(\sigma_2(\delta)) - \sigma_2(\delta) \leq f(\sigma_2) - \sigma_2 + (f_{\max} - 1)\hat{\sigma}_2(\delta)$. The budget constraint implies that $\frac{\hat{b}_2(\delta)}{R_2} \leq (f_{\max} - 1)\hat{\sigma}_2(\delta) + [(1 + C_{-1})R_{\max}^2A_0 + R_{\max}A_1]\delta$. Then, the borrowing constraint implies that $\hat{\sigma}_2(\delta) \leq \phi f_{\max}\hat{\sigma}_2(\delta) + \delta + (f_{\max} - 1)\hat{\sigma}_2(\delta) + [(1 + C_{-1})R_{\max}^2A_0 + R_{\max}A_1]\delta$, which can be rewritten as $\hat{\sigma}_2(\delta) = \frac{[1+(1+C_{-1})R_{\max}^2A_0+R_{\max}A_1]}{2-(1+\phi)f_{\max}}\delta$. We define

$$A_2 \equiv A_0[1 + (1 + C_{-1})R_{\max}^2A_0 + R_{\max}A_1].$$

Then, $(f_{\max} - 1)\hat{\sigma}_2(\delta) = A_2\delta$ and $\frac{b_2(\delta)}{R_2} \geq \frac{b_2}{R_2} - [(1 + C_{-1})R_{\max}^2A_0 + R_{\max}A_1 + A_2]\delta$. Similarly, the following claim is proven by induction. To simplify the notation, we define

$$A_{-1} \equiv \frac{C_{-1}A_0}{R_{\max}}.$$

Then, $A_1 = A_0(1 + R_{\max}A_0 + R_{\max}^2A_{-1})$ and $A_2 = A_0(1 + R_{\max}A_1 + R_{\max}^2A_0 + R_{\max}^3A_{-1})$.

Claim 1. Suppose the economy is in Stage 2 in period n (for $0 \leq n \leq \bar{n}$). Then,

$$\frac{\hat{b}_n(\delta)}{R_n} \leq \left(\sum_{i=0}^{n+1} R_{\max}^i A_{n-i}\right)\delta, \tag{32}$$

where A_n is defined by

$$A_n \equiv A_0 \left[1 + R_{\max} \left(\sum_{i=0}^{n+1} R_{\max}^i A_{n-i} \right) \right].$$

Proof. This claim holds true for $n = 0, 1, 2,$, as we see above. Suppose (32) holds true for $n = j$. Then, for $n = j + 1$, the budget constraint and borrowing constraint imply that

$$0 \leq (f_{\max} - 1)\hat{\sigma}_{j+1}(\delta) + R_{\max} \left(\sum_{i=0}^{j+1} R_{\max}^i A_{j-i} \right) \delta - \frac{\hat{b}_{j+1}(\delta)}{R_{j+1}}, \tag{33}$$

$$\hat{\sigma}_{j+1}(\delta) \leq \phi f_{\max} \hat{\sigma}_{j+1}(\delta) + \delta + \frac{\hat{b}_{j+1}(\delta)}{R_{j+1}}.$$

These equations imply

$$(f_{\max} - 1)\hat{\sigma}_{j+1}(\delta) \leq A_0 \left[1 + R_{\max} \left(\sum_{i=0}^{j} R_{\max}^i A_{j-i} \right) \right] \delta.$$

Defining $A_{j+1} \equiv A_0 \left[1 + R_{\max} \left(\sum_{i=0}^{j} R_{\max}^i A_{j-i} \right) \right]$, this is written as $(f_{\max} - 1)\hat{\sigma}_{j+1}(\delta) = A_{j+1}\delta$, and (33) implies that $\frac{\hat{b}_{j+1}(\delta)}{R_{j+1}} \leq A_{j+1}\delta + R_{\max} \left(\sum_{i=0}^{j} R_{\max}^i A_{j-i} \right) \delta = \left(\sum_{i=0}^{j+1} R_{\max}^i A_{j+1-i} \right) \delta$. Therefore, it has been shown that (32) holds true for $0 \leq n \leq \bar{n}$, as long as period n is in Stage 2. ☐

Now, as $t_2(\delta) - t_1(\delta) \leq \bar{n}$, it holds true that for $n \leq t_2(\delta) - t_1(\delta)$,

$$\frac{\hat{b}_n(\delta)}{R_n} \leq \Theta\delta,$$

where the constant, Θ, is defined by

$$\Theta \equiv \left(\sum_{i=0}^{\bar{n}+1} R_{\max}^i A_{\bar{n}-i} \right). \tag{34}$$

Stage 3: Now, we use subscript t instead of n. We can show that $t_2 - 1 \leq t_2(\delta) \leq t_2$, as follows. Define $B_t^{ce}(\delta) \equiv f(\sigma_t^{ce}, x_t) - \sigma_t^{ce} + \frac{b_t^{ce}(\delta)}{R_t}$ and $\frac{b_t^{ce}(\delta)}{R_t} \equiv \phi f(\sigma_t^{ce}, x_t) - \sigma_t^{ce} + \xi S_t + \delta$. Thus, $B_t^{ce}(\delta) = B_t^{ce} + \delta$ and $\frac{b_t^{ce}(\delta)}{R_t} = \frac{b_t^{ce}}{R_t} + \delta$. We know that $b_{t_2-1} < B_{t_2}^{ce}$ and

$$b_t > B_{t+1}^{ce}, \qquad \text{for } \; t \leq t_2 - 2. \tag{35}$$

The variables $b_t(\delta)$ and $B_{t+1}^{ce}(\delta)$ for $t_1(\delta) \leq t \leq t_2 - 3$ satisfy the following inequality:

$$\begin{aligned} b_t(\delta) - B_{t+1}^{ce}(\delta) &\geq b_t - B_{t+1}^{ce} - (\Theta + 1)\delta \\ &> \inf_{x \in \Lambda} (1 - \phi) f(\sigma_z(x), x) + \frac{b_{t+1}}{R_{t+1}} - B_{t+1}^{ce} - (\Theta + 1)\delta \\ &> \inf_{x \in \Lambda} (1 - \phi) f(\sigma_z(x), x) + \frac{B_{t+2}^{ce}}{R_{t+1}} - B_{t+1}^{ce} - (\Theta + 1)\delta \\ &> \inf_{x \in \Lambda} (1 - \phi) f(\sigma_z(x), x) - \nabla - (\Theta + 1)\delta > 0. \end{aligned}$$

The first inequality is due to $b_t(\delta) \geq b_t - \Theta\delta$ and $B_t^{ce}(\delta) = B_t^{ce} + \delta$. The second inequality is due to $b_t = \frac{b_{t+1}}{R_{t+1}} + f(\sigma_{t+1}, x_{t+1}) - \sigma_{t+1} \geq \frac{b_{t+1}}{R_{t+1}} + \inf(1-\phi)f(\sigma_{z,t}, x_t)$. The third inequality is due to (35). The fourth inequality is due to definition of ∇ in Assumption 3. The final inequality is due to (27). Thus, we have shown that the economy is in Stage 2 for $t \leq t_2 - 2$. Therefore, it must be the case that $t_2(\delta) = t_2 - 1$ or t_2. We evaluate $|V(b, x; S) - V(b, x; S + \delta/\xi)|$ for each case.

- **Case where $t_2(\delta) = t_2$:**

  In this case,

$$\begin{aligned}
\pi_{t_2}(\delta) &= f(\sigma_{t_2}^{ce}, x_{t_2}) - \sigma_{t_2}^{ce} + \frac{b_{t_2}^{ce}(\delta)}{R_{t_2}} - b_{t_2-1}(\delta) \\
&\leq f(\sigma_{t_2}^{ce}, x_{t_2}) - \sigma_{t_2}^{ce} + \frac{b_{t_2}^{ce}}{R_{t_2}} + \delta - b_{t_2-1} + \Theta\delta \\
&= \pi_{t_2} + (\Theta + 1)\delta.
\end{aligned}$$

  The inequality is due to $\frac{b_{t_2}^{ce}(\delta)}{R_{t_2}} = \frac{b_{t_2}^{ce}}{R_{t_2}} + \delta$ and $b_{t_2-1}(\delta) \geq b_{t_2-1} - \Theta\delta$. For $t \geq t_2 + 1$,

$$\begin{aligned}
\pi_t(\delta) &= f(\sigma_t^{ce}, x_t) - \sigma_t^{ce} + \frac{b_t^{ce}(\delta)}{R_t} - b_{t-1}^{ce}(\delta) \\
&= \pi_t + (1 - R_{t-1})\delta.
\end{aligned}$$

  In this case, using the fact that $t_2(\delta) \geq 0$,

$$\begin{aligned}
|V(b, x; S) - V(b, x; S + \delta/\xi)| &\leq \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \frac{m_t}{m_0} |\pi_t - \pi_t(\delta)| \right] \\
&\leq (\Theta + 1)\delta + \sum_{t=1}^{\infty} \frac{R_{\max} - 1}{(1 + r_{\min})^t} \delta \\
&= \left(\Theta + 1 + \frac{R_{\max} - 1}{r_{\min}} \right) \delta.
\end{aligned}$$

- **Case where $t_2(\delta) = t_2 - 1$:**

  In the original firm's problem with S, it holds true that $b_{t_2-2} > B_{t_2-1}^{ce}$ and $b_{t_2-1} \leq B_{t_2}^{ce}$, whereas in the modified firm's problem with $S + \delta/\xi$, it holds true that $b_{t_2-2}(\delta) \leq B_{t_2-1}^{ce}(\delta)$, where $b_{t_2-2}(\delta) \geq b_{t_2-2} - \Theta\delta$. In the original problem, the budget constraint with $\pi_{t_2-1} = 0$ and the borrowing constraint imply

$$(1 + \phi)f(\sigma_{t_2-1}, x_{t_2-1}) - \sigma_{t_2-1} = b_{t_2-2} - \xi S_{t_2-1}. \tag{36}$$

  In the modified problem, there exists $\hat{C}(b_{t_2-2})$, such that $0 < \hat{C}(b_{t_2-2}) \leq \Theta$ and

$$b_{t_2-2} - \hat{C}(b_{t_2-2})\delta = B_{t_2-1}^{ce}(\delta).$$

  This equation and the borrowing constraint $\sigma_{t_2-1}^{ce} = \phi f(\sigma_{t_2-1}^{ce}, x_{t_2-1}) + \xi S_{t_2-1} + \delta - b_{t_2-1}^{ce}(\delta)/R_{t_2-1}$ together imply that

$$(1 + \phi)f(\sigma_{t_2-1}^{ce}, x_{t_2-1}) - \sigma_{t_2-1}^{ce} = b_{t_2-2} - \xi S_{t_2-1} - (\hat{C}(b_{t_2-2}) + 1)\delta. \tag{37}$$

The derivative of function $(1+\phi)f(\sigma,x)-2\sigma$ satisfies

$$-2 < (1+\phi)\frac{\partial}{\partial\sigma}f(\sigma,x) - 2 < (1+\phi)f_{\max} - 2 < 0. \tag{38}$$

The rightmost inequality is due to $f_{\max} = \eta/\phi$ and $\frac{\eta}{2-\eta} < \phi$. Then, because $(1+\phi)f(\sigma,x) - 2\sigma$ is concave and continuously differentiable with respect to σ, the equations (36) and (37), together with inequality (38), imply that

$$\sigma^{ce}_{t_2-1} - \frac{\hat{C}(b_{t_2-2})+1}{2-(1+\phi)f_{\max}}\delta \le \sigma_{t_2-1} \le \sigma^{ce}_{t_2-1} - \frac{\hat{C}(b_{t_2-2})+1}{2}\delta.$$

The fact that $0 < \hat{C}(b_{t_2-2}) \le \Theta$ implies that

$$\sigma^{ce}_{t_2-1} - \Phi\delta \le \sigma_{t_2-1} \le \sigma^{ce}_{t_2-1} - \frac{1}{2}\delta, \tag{39}$$

where $\Phi \equiv \frac{\Theta+1}{2-(1+\phi)f_{\max}}$. This inequality and the concavity and differentiability of $f(\sigma,x)$ imply that

$$\frac{b^{ce}_{t_2-1}(\delta)}{R_{t_2-1}} + (\eta\Phi+1)\delta < \frac{b_{t_2-1}}{R_{t_2-1}} < \frac{b^{ce}_{t_2-1}(\delta)}{R_{t_2-1}} + (\Phi-1)\delta, \tag{40}$$

because $b_{t_2-1}/R_{t_2-1} = \phi f(\sigma_{t_2-1}, x_{t_2-1}) - \sigma_{t_2-1} + \xi S_{t_2-1}$. Now, we evaluate $\pi_t(\delta)$ for $t = t_2-1, t_2, t_2+1, \cdots$. We know $\pi_{t_2-1} = 0$ in the original problem. The inequalities (39) and (40), together with $b_{t_2-2}(\delta) \ge b_{t_2-2} - \Theta\delta$ imply that

$$\begin{aligned}
\pi_{t_2-1}(\delta) &\le f(\sigma^{ce}_{t_2-1}, x_{t_2-1}) - \sigma^{ce}_{t_2-1} + \frac{b^{ce}_{t_2-1}(\delta)}{R_{t_2-1}} - b_{t_2-2} + \Theta\delta \\
&\le f(\sigma_{t_2-1}+\Phi\delta, x_{t_2-1}) - \sigma_{t_2-1} - \frac{1}{2}\delta + \frac{b_{t_2-1}}{R_{t_2-1}} - (\eta\Phi+1)\delta - b_{t_2-2} + \Theta\delta \\
&< \left[(f_{\max}-\eta)\Phi + \Theta - \frac{3}{2}\right]\delta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi_{t_2}(\delta) &= f(\sigma^{ce}_{t_2}, x_{t_2}) - \sigma^{ce}_{t_2} + \frac{b^{ce}_{t_2}(\delta)}{R_{t_2}} - b^{ce}_{t_2-1}(\delta) \\
&\le f(\sigma^{ce}_{t_2}, x_{t_2}) - \sigma^{ce}_{t_2} + \frac{b^{ce}_{t_2}}{R_{t_2}} + \delta - b_{t_2-1} + R_{t_2-1}(\Phi-1)\delta \\
&= \pi_{t_2} + [1 + R_{t_2-1}(\Phi-1)]\delta.
\end{aligned}$$

For $t \ge t_2+1$, the same result holds as in the case where $t_2(\delta) = t_2$:

$$\pi_t(\delta) = \pi_t + (1-R_{t-1})\delta.$$

Define $\Psi \equiv \sup_{x\in\Lambda} |1 + R(x)(\Phi-1)|$. Then, using $t_2 - 1 \ge 0$,

$$\begin{aligned}
|V(b,x;S) - V(b,x;S+\delta/\xi)| &\le \mathbb{E}_0\left[\sum_{t=0}^{\infty}\frac{m_t}{m_0}|\pi_t - \pi_t(\delta)|\right] \\
&\le \left[(f_{\max}-\eta)\Phi + \Theta - \frac{3}{2}\right]\delta + \frac{\Psi}{1+r_{\min}}\delta + \sum_{t=2}^{\infty}\frac{R_{\max}-1}{(1+r_{\min})^t}\delta \\
&= \left[(f_{\max}-\eta)\Phi + \Theta - \frac{3}{2} + \frac{\Psi}{1+r_{\min}} + \frac{R_{\max}-1}{(1+r_{\min})r_{\min}}\right]\delta.
\end{aligned}$$

The above analysis of Stages 1–3 implies that there exists a positive number C that does not depend on (ε, δ, S), such that for a sufficiently small and positive number δ that satisfies (27),

$$|V(b, x; S + \delta/\xi) - V(b, x; S)| < C\delta.$$

This inequality implies that, for any small $\varepsilon > 0$, there exists $\delta \leq \frac{\varepsilon}{C}$, such that $|V(b, x; S') - V(b, x; S)| < \varepsilon$ if $|S' - S| < \delta$. Thus, $V(b, x; S)$ is continuous with respect to S. ☐

Given the continuity of V, it is obvious that T is continuous with respect to S; that is, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|TS' - TS| < \varepsilon$ if $|S' - S| < \delta$. The equicontinuity of $T(G)$ is also shown as follows.

Lemma 11. *$T(G)$ is an equicontinuous family; that is, for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $TS(x) \in T(G)$, $|TS(x') - TS(x)| < \varepsilon$ if $|x' - x| < \delta$.*

Proof. Pick a small $\varepsilon\,(> 0)$ arbitrarily. We fix ε throughout this proof. As TS is defined as $TS(x) = \max_b \mathbb{E}\left[\frac{m'}{m(x)} V(b, x', S)\right] + \frac{b}{R(x)}$, the continuity of V with respect to x (see Proposition 1) implies that there exists $\delta > 0$ such that $|TS(x') - TS(x)| < \frac{\varepsilon}{2}$, if $|x' - x| < \delta$. Note that the maximum value of δ that satisfies the above condition must depend on $S \in G$. Define $\delta(S)$ by

$$\delta(S) \equiv \sup\{\delta |\ |TS(x') - TS(x)| < \tfrac{\varepsilon}{2}, \text{ if } |x' - x| < \delta\}.$$

Pick an arbitrary sequence of positive real numbers $\{\delta_n\}_{n=0}^{\infty}$ that satisfies $0 < \delta_{n+1} < \delta_n$ and $\lim_{n\to\infty} \delta_n = 0$. Define

$$G_n \equiv \{S | \delta(S) \geq \delta_n, S \in G\}.$$

Obviously,

$$G_n \subseteq G_{n+1} \subseteq G, \qquad \text{and} \qquad \lim_{n\to\infty} G_n = G.$$

Now, the following claim holds.

Claim 2. For all $\delta > 0$, there exists n, such that for all $S \in G$, there exists $S_n \in G_n$, such that $|S - S_n| \leq \delta$.

Proof. Suppose that this claim is false. Then, there exists $\delta > 0$, such that for all $n > 0$, there exists $S_n^c \in G$, such that $|S_n^c - S_n| > \delta$ for all $S_n \in G_n$. We denote the set of S_n^c by $G_n^c(\delta)$. The assumption implies that $G_n^c(\delta) \neq \emptyset$, for all n. Obviously, $G_n \cap G_n^c(\delta) = \emptyset$, $G_n \cup G_n^c(\delta) \subseteq G$, and $G_{n+1}^c(\delta) \subseteq G_n^c(\delta) \subseteq G$. Taking the limit of $n \to \infty$ implies that $\lim_{n\to\infty} G_n^c(\delta) \neq \emptyset$, which contradicts $\lim_{n\to\infty} G_n = G$. ☐

The continuity of T implies that there exists δ_1 such that $|TS'-TS| < \frac{\varepsilon}{4}$ if $|S'-S| \leq \delta_1$. The above claim implies that there exists n, such that for all $S \in G$, there exists $S_n \in G_n$ that satisfy $|S - S_n| \leq \delta_1$. By definition, $|TS_n(x') - TS_n(x)| \leq \frac{\varepsilon}{2}$ if $|x' - x| < \delta_n$. Now, we show the equicontinuity of $T(G)$. For a given $\varepsilon(> 0)$, we choose δ_1 and G_n as above. Given $(\varepsilon, \delta_1, \delta_n)$, it holds that for all $S \in G$, there exists $S_n \in G_n$ such that the following inequality holds for any x and x' $(\in \Lambda)$ that satisfy $|x - x'| \leq \delta_n$:

$$\begin{aligned}
|TS(x') - TS(x)| &= |TS(x') - TS_n(x') - TS(x) + TS_n(x) + TS_n(x') - TS_n(x)| \\
&\leq |TS(x') - TS_n(x')| + |TS(x) - TS_n(x)| + |TS_n(x') - TS_n(x)| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, it has been shown that $T(G)$ is equicontinuous. □

As the mapping $T : G \to G$ is continuous and the family $T(G)$ is equicontinuous, given that G is nonempty, closed, bounded, and convex, the Schauder fixed point theorem implies that T has a fixed point in G.

C Proof of Proposition 9

As Lemma 4 holds in the deterministic case, it is sufficient to show that $\bar{B}_{c,t}$ is strictly smaller than $\bar{B}_{z,t} = b_{z,t-1}$ to prove Proposition 9.

Define $\Delta_t \equiv f_t(\sigma_{z,t}) - \sigma_{z,t}$. We also define $V_t^{(i)}(b_{t-1})$, for $i = 1, 2$, by $V_t^{(i)}(b_{t-1}) = W_t^{(i)}(b_{t-1}) - \frac{1+r_{t-1}}{R_{t-1}} b_{t-1}$. They are written as follows when the borrowing constraint is (19) for period t and $t + 1$:

$$\begin{aligned}
V_t^{(1)}(b_{t-1}) &= \Delta_t^z - b_{t-1} + \frac{b_{z,t}}{R_t} + \frac{1}{1+r_t} V_{t+1}(b_{z,t}) = \Delta_t^z - b_{t-1} + \frac{b_{z,t}}{R_t}, \\
V_t^{(2)}(b_{t-1}) &= \frac{1}{1+r_t} V_{t+1}(R_t\{b_{t-1} - \Delta_t^z\}), \\
V_t(b_{t-1}) &= \max\ \{V_t^{(1)}(b_{t-1}),\ V_t^{(2)}(b_{t-1})\}.
\end{aligned}$$

We can show the following lemma.

Lemma 12. *If* $V_t^{(1)}(b_{t-1}) < V_t^{(2)}(b_{t-1})$, *then* $V_{t+j}(b_{t+j-1}) = V_{t+j}^{(2)}(b_{t+j-1})$ *for all* $j \in \{1, 2, \cdots, J\}$, *where* $b_{t+j-1} = R_{t+j-1}\{b_{t+j-2} - \Delta_{t+j-1}^z\}$ *and* J *is defined by* $J = \min\{j | b_{t+j} < B_{z,t+j+1}\}$.

Proof. Proof is by contradiction. Suppose, on the contrary, $V_t^{(1)}(b_{t-1}) < V_t^{(2)}(b_{t-1})$ and $V_{t+1}(b_t) = V_{t+1}^{(1)}(b_t)$, where $b_t = R_t\{b_{t-1} - \Delta_t^z\}$. It is calculated that

$$\begin{aligned}
V_t^{(2)}(b_{t-1}) &= \frac{1}{1+r_t} V_{t+1}(R_t\{b_{t-1} - \Delta_t^z\}) = \frac{1}{1+r_t} V_{t+1}^{(1)}(R_t\{b_{t-1} - \Delta_t^z\}) \\
&= \frac{1}{1+r_t}\left[\Delta_{t+1}^z - R_t\{b_{t-1} - \Delta_t^z\} + \frac{b_{z,t+1}}{R_{t+1}}\right] = \frac{R_t}{1+r_t}\left[\Delta_t^z - b_{t-1} + \frac{b_{z,t}}{R_t}\right] \\
&= \frac{R_t}{1+r_t} V^{(1)}(b_{t-1}) < V_t^{(1)}(b_{t-1}).
\end{aligned}$$

The last inequality is due to $\frac{R_t}{1+r_t} < 1$. This inequality contradicts the assumption. Thus, it has been shown that if $V_t^{(1)}(b_{t-1}) < V_t^{(2)}(b_{t-1})$, then $V_{t+1}^{(2)}(b_t) \geq V_{t+1}^{(1)}(b_t)$ and $V_{t+1}(b_t) = V_{t+1}^{(2)}(b_t)$. The above argument continues to hold in period $t+j$ as long as the borrowing constraint is (19), which means that $j = 1, 2, \cdots, J$. □

This lemma implies that if $V_t^{(1)}(b_{t-1}) < V_t^{(2)}(b_{t-1})$, then $V_{t+j}^{(2)}(b_{t+j-1}) = \frac{1}{1+r_{t+j}} V_{t+j+1}^{(2)}(b_{t+j})$ for $j = 1, 2, \cdots, J$, and, therefore, $V_t^{(2)}(b_{t-1}) = \tilde{V}_t^{(2)}(b_{t-1})$, where

$$\tilde{V}_t^{(2)}(b_{t-1}) \equiv \frac{1}{\prod_{j=0}^{J}(1+r_{t+j})} V_{t+J+1}(b_{t+J}),$$

where J and b_{t+J} are defined in the above lemma.

First, we restrict our attention to the case where debt b_{t-1} takes discrete values. We assume that $b_{t-1} \in \Gamma_{t-1}$, where $\Gamma_{t-1} = \{b_{t-1}^{[n]} | n = 1, 2, \cdots, \infty\}$, where $b_{t-1}^{[n]}$ is specified below shortly. Under this assumption, we show that there exists n^*, such that $b_{t-1}^{[n^*]}$ is the closest to $\bar{B}_{c,t}$ among all the elements of Γ_{t-1}. We then proceed to the case where b_{t-1} takes a continuous value. We define

$$B_t^S = R_t \xi S_t.$$

The definition of $b_{t-1}^{[n]}$ is the debt that becomes $\frac{B_{t+n-1}^S}{R_{t+n-1}}$ in period $t+n-1$ if the firm repays Δ_{t+j}^z in period $t+j$ for $j = 0, 1, \cdots, n-1$. Thus, it is written as follows:

$$b_{t-1}^{[n]} = \sum_{j=0}^{n-1} \frac{\Delta_{t+j}^z}{\prod_{i=0}^{j-1} R_{t+i}} + \frac{B_{t+n-1}^S}{\prod_{j=0}^{n-1} R_{t+j}},$$

where we define $\prod_{j=0}^{-1} R_{t+j} = 1$. The upper limit of the debt is also written as follows:

$$b_{z,t-1} = \sum_{j=0}^{\infty} \frac{\Delta_{t+j}^z}{\prod_{i=0}^{j-1} R_{t+i}}.$$

Now, let us consider the discrete case where $b_{t-1} \in \Gamma_{t-1}$. Then, $V_t^{(1)}(b_{t-1}^{[n]}) = \Delta_t^z - b_{t-1}^{[n]} + \frac{b_{z,t}}{R_t} = b_{z,t-1} - b_{t-1}^{[n]}$ is written as follows:

$$V_t^{(1)}(b_{t-1}^{[n]}) = \frac{1}{\prod_{j=0}^{n-1} R_{t+j}} [b_{z,t+n-1} - B_{t+n-1}^S].$$

$\tilde{V}_t^{(2)}(b_{t-1}^{[n]})$ is written as follows:

$$\tilde{V}_t^{(2)}(b_{t-1}^{[n]}) = \frac{1}{\prod_{j=0}^{n-1}(1+r_{t+j})} V_{t+n}(B_{t+n-1}^S).$$

Therefore,

$$\frac{V_t^{(1)}(b_{t-1}^{[n]})}{\tilde{V}_t^{(2)}(b_{t-1}^{[n]})} = \left[\frac{b_{z,t+n-1} - B_{t+n-1}^S}{V_{t+n}(B_{t+n-1}^S)}\right] \prod_{j=0}^{n-1} \left(\frac{1+r_{t+j}}{R_{t+j}}\right).$$

As it is always the case that $B_t^S = \xi R_t S_t \leq \xi R_t \omega_t$ and $V_t(b) \leq \omega_t$, the above equation implies that

$$\frac{V_t^{(1)}(b_{t-1}^{[n]})}{\tilde{V}_t^{(2)}(b_{t-1}^{[n]})} \geq \left[\frac{b_{z,t+n-1} - \xi R_{t+n-1}\omega_{t+n-1}}{\omega_{t+n}}\right] \prod_{j=0}^{n-1} \left(\frac{1+r_{t+j}}{R_{t+j}}\right). \tag{41}$$

Then, it is immediate from (9), (22), and (41) that there exists a finite integer n^* (> 0), such that $V_t^{(1)}(b_{t-1}^{[n]}) > \tilde{V}_t^{(2)}(b_{t-1}^{[n]})$, for all $n \geq n^*$. Because the parameters are chosen such that $B_t^S < b_{z,t}$ (see (9)), it is immediate from the definition of $b_{t-1}^{[n]}$ that $b_{t-1}^{[n^*]} < b_{z,t-1}$. Thus, we have established that the threshold value $b_{t-1}^{[n^*]}$ exists and is strictly smaller than $b_{z,t-1}$ in the discrete case where $b_{t-1} \in \Gamma_{t-1}$.

Let us proceed to the continuous case where b_{t-1} is not restricted to an element of Γ_{t-1}, but is a positive real number. We know from Lemma 4 that $\bar{B}_{c,t} \leq b_{z,t-1}$. We prove $\bar{B}_{c,t} < b_{z,t-1}$ by contradiction.

Suppose that $\bar{B}_{c,t} = b_{z,t-1}$. We know that $V_t^{(2)}(b_{t-1})$ is continuous and decreasing in b_{t-1} and that $V_t^{(1)}(b_{t-1})$ is a linear function of b_{t-1}. These facts imply that there must exist a small positive number ε (> 0) such that for all $b_{t-1} \in [b_{z,t-1} - \varepsilon, b_{z,t-1})$, $V_t^{(1)}(b_{t-1}) \leq \tilde{V}_t^{(2)}(b_{t-1})$. However, this result obviously contradicts $V_t^{(1)}(b_{t-1}^{[n]}) > \tilde{V}_t^{(2)}(b_{t-1}^{[n]})$ for all $n \geq n^*$, because there exists $n (> n^*)$, such that $b^{[n]} \in [b_{z,t-1} - \varepsilon, b_{z,t-1})$, as $\lim_{n\to\infty} b_{t-1}^{[n]} = b_{z,t-1}$. Thus, it cannot hold that $\bar{B}_{c,t} = b_{z,t-1}$. Therefore, $\bar{B}_{c,t} < b_{z,t-1}$.

D The BGP

The detrended variables of the BGP are determined by the following system of equations:

$$
\begin{aligned}
&A = Y^{1-\eta},\\
&Y = K^{\alpha} L_p^{1-\alpha} = A K^{\alpha\eta} L_p^{(1-\alpha)\eta},\\
&C + (g^E - 1 + \rho)K = Y,\\
&w = \frac{\gamma C}{1-L},\\
&w = \frac{(1-\alpha)\eta(1+\phi\mu)}{1+\mu}\frac{Y}{L_p},\\
&r^K = \frac{\alpha\eta(1+\phi\mu)}{1+\mu}\frac{Y}{K},\\
&1 + r = \frac{g^E}{\beta},\\
&R = 1 + (1-\tau)r,\\
&\mu = 1 - \frac{\beta}{g^E}R,\\
&1 = \frac{\beta}{g^E}\left(r^K + 1 - \rho\right),\\
&g = \left[1 + \kappa(L - L_p)\right],\\
&\pi = Y - wL - r^K K - B\left[1 - \frac{g^E}{R}\right],\\
&V_n = \frac{\pi}{1-\beta},\\
&S = \frac{\beta}{g}V_n + \frac{g^{E-1}}{R}B,\\
&wL + r^K K = \phi Y + \left[g\xi S - \frac{g^E B}{R}\right],\\
&\frac{\beta}{g}\kappa V_n = (1+\mu)w - \kappa g^{E-1}\frac{B}{R} - \mu\kappa\left[\xi S - \frac{g^{E-1}B}{R}\right].
\end{aligned}
$$

where μ is the Lagrange multiplier associated with the borrowing constraint and B is the total amount of corporate lending, which satisfies $B = b$, where b is the outstanding debt for a firm. There are 16 equations for 16 unknowns: $\{g, A, Y, K, L, L_p, C, w, \mu, r, r^K, R, \pi, B, V_n, S\}$.

The following condition must be satisfied on the BGP:

$$\xi S > g^{E-1}\frac{B}{R}.$$

Then, the variables for debt-ridden firms on the BGP, $\{k_z, l_z, b_z, \frac{\mu_z}{\lambda_z}, \frac{\nu_z}{\lambda_z}\}$, are calculated as follows:

$$
\begin{aligned}
w &= \frac{(1-\alpha)\eta\left(1+\phi\frac{\mu_z}{\lambda_z}\right)}{1+\frac{\mu_z}{\lambda_z}} A k_z^{\alpha\eta} l_z^{(1-\alpha)\eta-1}, \\
r^K &= \frac{\alpha\eta\left(1+\phi\frac{\mu_z}{\lambda_z}\right)}{1+\frac{\mu_z}{\lambda_z}} A k_z^{\alpha\eta-1} l_z^{(1-\alpha)\eta}, \\
& w l_z + r^K k_z = \phi A k_z^{\alpha\eta} l_z^{(1-\alpha)\eta}, \\
0 &= A k_z^{\alpha\eta} l_z^{(1-\alpha)\eta} - w l_z - r^K k_z + g^{E-1}\frac{b_z}{R} - b_z, \\
0 &= \left(1+\frac{\mu_z}{\lambda_z}\right) w - \kappa g^{E-1}\frac{b_z}{R} - \frac{\nu_z}{\lambda_z}.
\end{aligned}
$$

The following conditions must be satisfied on the BGP:

$$
\begin{aligned}
& \frac{\nu_z}{\lambda_z} > 0, \\
& \xi S < g^{E-1}\frac{b_z}{R} < S.
\end{aligned}
$$

E Detrending for a firm's problems

We detrend the variables as follows: $A_t = \tilde{A}_t N_{t-1}^{(1-\eta)E}$, $Y_t = \tilde{Y}_t N_{t-1}^E$, $C_t = \tilde{C}_t N_{t-1}^E$, $K_t = \tilde{K}_t N_t^E$, $k_{z,t} = \tilde{k}_{z,t} N_{t-1}^{E-1}$, $k_{n,t} = \tilde{k}_{n,t} N_{t-1}^{E-1}$, $l_{z,t} = \tilde{l}_{z,t}/N_{t-1}$, $l_{n,t} = \tilde{l}_{n,t}/N_{t-1}$, $w_t = \tilde{w}_t N_{t-1}^E$, $\pi_t = N_{t-1}^{E-1}\tilde{\pi}_t$, $b_{n,t} = \tilde{b}_{n,t} N_{t-1}^{E-1}$, $b_{z,t} = \tilde{b}_{z,t} N_{t-1}^{E-1}$, $V_{n,t} = \tilde{V}_{n,t} N_{t-1}^{E-1}$ $z_t = Z_t/N_{t-1}$, and $g_t = N_t/N_{t-1}$.

Detrending TFP: Suppose that $TFP = T\tilde{F}P \cdot N^a$. The equation $Y = TFP \cdot K^\alpha L^{1-\alpha}$ implies that

$$
\tilde{Y} N^{\frac{1-\eta}{(1-\alpha)\eta}} = T\tilde{F}P \cdot N^a \tilde{K}^\alpha N^{\frac{(1-\eta)\alpha}{(1-\alpha)\eta}} L^{1-\alpha},
$$

which implies that $a = \frac{1-\eta}{\eta}$ and, thus,

$$
TFP = T\tilde{F}P \cdot N^{\frac{1-\eta}{\eta}}.
$$

F Transition dynamics with $z > 0$

In this appendix, we describe the transition dynamics in the case where the economy is initially on the BGP and the z proportion of firms are suddenly imposed the maximum debt b_z at time 10, where $0 < z < 1$. The economy eventually converges to the BGP. All agents have perfect foresight on the path after the one-time buildup of debt, which is the only unexpected event. In this setting, we can apply a deterministic simulation with an occasionally binding borrowing constraint by using Dynare (see Adjemian et al., 2011). In our model, the borrowing constraint is always binding both in transition and on the BGP.

The equilibrium is calculated by solving a full nonlinear system of simultaneous equations using a modified Newton–Raphson algorithm. The details of the algorithm can be found in Juillard (1996). This algorithm solves 31×300 simultaneous equations, where 31 is the number of endogenous variables and 300 is the number of simulation periods.

Altogether, 31 variables—$\{\tilde{A}_t, \tilde{Y}_t, \tilde{K}_t, \tilde{C}_t, L_t, R_t, \tilde{l}_{n,p,t}, \tilde{l}_{z,p,t}, \tilde{l}_{z,t}, \tilde{l}_{n,t}, \tilde{k}_{z,t}, \tilde{k}_{n,t}, \tilde{b}_{z,t}, \tilde{b}_{n,t}, \tilde{y}_{z,t}, \tilde{y}_{n,t}, g_t, r_t, r_t^K, \tilde{w}_t, z_t, \tilde{V}_{n,t}, \tilde{S}_t, \tilde{\pi}_{n,t}, \tilde{\pi}_{z,t}, \frac{\mu_{z,t}}{\lambda_{z,t}}, \frac{\mu_{n,t}}{\lambda_{n,t}}, \frac{\lambda_{l,n,t}}{\lambda_{n,t}}, \frac{\lambda_{l,z,t}}{\lambda_{z,t}}, \lambda_{\pi,n,t}, \lambda_{n,t}\}$—are calculated from the following 31 equations:

$$
\begin{aligned}
\tilde{A}_t &= \tilde{Y}_t^{1-\eta}, \\
\tilde{Y}_t &= \{z_t \tilde{y}_{z,t}^{\eta} + (1 - z_t)\tilde{y}_{n,t}^{\eta}\}^{\frac{1}{\eta}}, \\
L_t &= z_t \tilde{l}_{z,t} + (1 - z_t)\tilde{l}_{n,t}, \\
\tilde{K}_{t-1} &= z_t \tilde{k}_{z,t} + (1 - z_t)\tilde{k}_{n,t}, \\
\tilde{C}_t + g_t^E \tilde{K}_t - (1 - \rho)\tilde{K}_{t-1} &= \tilde{Y}_t, \\
\tilde{w}_t &= \frac{\gamma \tilde{C}_t}{1 - L_t}, \\
1 &= \frac{\beta \tilde{C}_t}{g_t^E \tilde{C}_{t+1}} \left(r_{t+1}^K + 1 - \rho \right), \\
1 + r_t &= \frac{g_t^E}{\beta} \frac{\tilde{C}_{t+1}}{\tilde{C}_t}, \\
R_t &= 1 + (1 - \tau) r_t, \\
g_t &= \left[1 + \kappa \left(1 - z_t\right) \left(\tilde{l}_{n,t} - \tilde{l}_{n,p,t} \right) \right], \\
z_t &= g_t z_{t+1}, \\
\tilde{S}_t &= g_t^{E-1} \left[\frac{\tilde{V}_{n,t+1}}{1 + r_t} + \frac{\tilde{b}_{n,t}}{R_t} \right],
\end{aligned}
$$

equations for both normal firms and debt-ridden firms, $i = n, z$:

$$
\begin{aligned}
\tilde{w}_t &= \frac{(1 - \alpha)\eta \left(1 + \phi \frac{\mu_{i,t}}{\lambda_{i,t}} \right)}{1 + \frac{\mu_{i,t}}{\lambda_{i,t}}} \tilde{A}_t \tilde{k}_{i,t}^{\alpha\eta} \tilde{l}_{i,p,t}^{(1-\alpha)\eta - 1}, \\
r_t^K &= \frac{\alpha\eta \left(1 + \phi \frac{\mu_{i,t}}{\lambda_{i,t}} \right)}{1 + \frac{\mu_{i,t}}{\lambda_{i,t}}} \tilde{A}_t \tilde{k}_{i,t}^{\alpha\eta - 1} \tilde{l}_{i,p,t}^{(1-\alpha)\eta}, \\
\tilde{y}_{i,t} &= \tilde{k}_{i,t}^{\alpha} \tilde{l}_{i,p,t}^{1-\alpha}, \\
\tilde{\pi}_{i,t} &= \tilde{A}_t \tilde{k}_{i,t}^{\alpha\eta} \tilde{l}_{i,p,t}^{(1-\alpha)\eta} - \tilde{w}_t \tilde{l}_{i,t} - r_t^K \tilde{k}_{i,t} + \frac{\tilde{b}_{i,t}}{R_t} \left[1 + \kappa \left(\tilde{l}_{i,t} - \tilde{l}_{i,p,t} \right) \right] g_t^{E-1} - \tilde{b}_{i,t-1},
\end{aligned}
$$

equations for normal firms only:

$$\tilde{l}_{n,p,t} \leq \tilde{l}_{n,t}, \quad \frac{\lambda_{l,n,t}}{\lambda_{n,t}}\left(\tilde{l}_{n,t} - \tilde{l}_{n,p,t}\right) = 0,$$

$$\frac{\beta \tilde{C}_t}{g_t \tilde{C}_{t+1}} \frac{\kappa \tilde{V}_{n,t+1}}{\lambda_{n,t}} = \left(1 + \frac{\mu_{n,t}}{\lambda_{n,t}}\right) \tilde{w}_t - \kappa g_t^{E-1} \frac{\tilde{b}_{n,t}}{R_t} - \frac{\lambda_{l,n,t}}{\lambda_{n,t}} - \kappa \frac{\mu_{n,t}}{\lambda_{n,t}} \max\left\{\xi \tilde{S}_t - \frac{g_t^{E-1}\tilde{b}_{n,t}}{R_t},\ 0\right\},$$

$$\begin{cases} \text{if } \xi \tilde{S}_t - \frac{g_t^{E-1}\tilde{b}_{n,t}}{R_t} \leq 0, & 1 = \dfrac{\beta \tilde{C}_t}{g_t^E \tilde{C}_{t+1}} \dfrac{\lambda_{n,t+1}}{\lambda_{n,t}} R_t, \\ \text{if } \xi \tilde{S}_t - \frac{g_t^{E-1}\tilde{b}_{n,t}}{R_t} > 0, & 1 - \dfrac{\mu_{n,t}}{\lambda_{n,t}} = \dfrac{\beta \tilde{C}_t}{g_t^E \tilde{C}_{t+1}} \dfrac{\lambda_{n,t+1}}{\lambda_{n,t}} R_t, \end{cases}$$

$$\begin{cases} \text{if } \mu_{n,t} > 0, & \tilde{w}_t \tilde{l}_{n,t} + r_t^K \tilde{k}_{n,t} = \phi \tilde{A}_t \tilde{k}_{n,t}^{\alpha\eta} l_{n,p,t}^{(1-\alpha)\eta} + \left[1 + \kappa\left(\tilde{l}_{n,t} - \tilde{l}_{n,p,t}\right)\right] \max\left\{\xi \tilde{S}_t - \frac{g_t^{E-1}\tilde{b}_{n,t}}{R_t},\ 0\right\}, \\ \text{if } \mu_{n,t} \leq 0, & \mu_{n,t} = 0, \end{cases}$$

$$\tilde{V}_{n,t} = \tilde{\pi}_{n,t} + \frac{\beta \tilde{C}_t}{g_t \tilde{C}_{t+1}} \left[1 + \kappa\left(\tilde{l}_{n,t} - \tilde{l}_{n,p,t}\right)\right] \tilde{V}_{n,t+1},$$

$$\begin{cases} \text{if } \lambda_{\pi,n,t} \leq 0, & \lambda_{n,t} = 1, \\ \text{if } \lambda_{\pi,n,t} > 0, & \tilde{\pi}_{n,t} = 0, \end{cases}$$

$$\lambda_{n,t} = 1 + \lambda_{\pi,n,t},$$

equations for debt-ridden firms only:

$$\tilde{l}_{z,t} = \tilde{l}_{z,p,t},$$

$$\tilde{\pi}_{z,t} = 0,$$

$$\tilde{w}_t \tilde{l}_{z,t} + r_t^K \tilde{k}_{z,t} = \phi \tilde{A}_t \tilde{k}_{z,t}^{\alpha\eta} l_{z,p,t}^{(1-\alpha)\eta} \implies \frac{\mu_{z,t}}{\lambda_{z,t}} = \frac{\eta - \phi}{\phi(1-\eta)},$$

$$\frac{\lambda_{l,z,t}}{\lambda_{z,t}} = \left(1 + \frac{\mu_{z,t}}{\lambda_{z,t}}\right) \tilde{w}_t - \kappa g_t^{E-1} \frac{\tilde{b}_{z,t}}{R_t} > 0.$$

The equilibrium variables must satisfy the following conditions:

$$\frac{g_t^{E-1}\tilde{b}_{n,t}}{R_t} < \xi \tilde{S}_t,$$

$$\xi \tilde{S}_t < \frac{g_t^{E-1}\tilde{b}_{z,t}}{R_t} < \tilde{S}_t.$$

G Calibration, data, and parameter region

Table 2 reports the values of the calibrated parameters. First, the parameters β, ρ, and η are common values to all countries. We set the discount factor β to 0.98, depreciation rate ρ to 0.1, and the parameter for the aggregation function η to 0.7. These are standard settings for an annual model in the literature.

Second, we calibrate the country-specific parameters and some BGP values. The tax advantage τ is set to 0.3 for Japan and 0.35 for the United States and the EU. These numbers are the corporate income tax rates in Japan and the United States, respectively. The share of labor in production (α), total labor supply on the BGP (L), and the growth rate of TFP on the BGP (g_{TFP}) are set from the data. Then, g_{TFP} is defined by $g_{TFP} = g^{\frac{1-\eta}{\eta}}$, as we see in Appendix E, and L is set to the ratio of average annual hours worked per person employed to total hours.

The sample period is from 1981 to 2008 for Japan, from 1997 to 2015 for the United States, and from 1997 to 2014 for EU. We assume that the economy is on the BGP before a financial crisis. In the case of Japan, the financial crisis starts in 1990, while in the case of the United States and EU, it starts in 2009. Then, α, L, and g_{TFP} in Japan are taken as the average during 1980–1990 from the JIP database,[20] and for the United States and EU they are taken as the average during 1997–2007. For the United States, L is from the Penn World Tables and the others are from Fernald (2012). For the EU, L and g_{TFP} are taken from the European Commission's Annual macro-economic database constructed by Havik et al. (2014), and the value of α is taken from Havik et al. (2014). When we calculate the variables on the BGP, L and g_{TFP} are given by the data. Hence, the inverse of the elasticity of labor supply γ and efficiency of R&D κ are determined endogenously in the system of the BGP, which is given in Appendix D.

Lastly, the parameters (ϕ, ξ, z) are identified by the simulated least squares criterion; that is, they are determined as the solution to the following minimization of the residual sum of squares (RSS) using a grid search method:

$$\min_{\phi,\ \xi,\ z} (\mathbf{X}_t - \hat{\mathbf{X}}_t)' \, (\mathbf{X}_t - \hat{\mathbf{X}}_t)$$

subject to

$$0 < \phi < \eta, \quad 0 < \xi < 1, \quad \frac{\lambda_{lz}}{\lambda_z} > 0, \quad \xi S < g^{E-1}\frac{b_z}{R} < S, \quad b_n > 0,$$

where $\mathbf{X}_t$ is the observed variables vector and $\hat{\mathbf{X}}_t$ is the simulation-generated variables vector. $\mathbf{X}_t$ = [TFP growth rate; per caita real output growth rate]. We extend the observed variables to 2025 by extrapolating the average growth rates in 2011–2015 for the United States and 2011–2014 for the EU. The simulation-generated variables, $\hat{\mathbf{X}}_t = [TFP_t/TFP_{t-1} - 1;\ Y_t/Y_{t-1} - 1]$, are calculated by the method presented in Appendix F, taking $(\gamma, \kappa, \phi, \xi, \chi, z)$ as the given exogenous parameters, where the debt shock hits the economy in period 10, i.e., $z_t = 0$ for $t \leq 9$ and $z_{10} = z$. Period 10 corresponds to 1990 for Japan and 2009 for the United States and EU. Minimization is done for the period 1991–2008 in the case of Japan, and for 2010–2025 in the case of the United States and EU.

We restrict the domain of parameters (ϕ, ξ) for the above minimization to the region that enables all constraints being satisfied on the BGP. Figure 8 denotes the domain of

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[20] We exclude data from 1987–1989, which was the period of abnormal boom called the "bubble period."

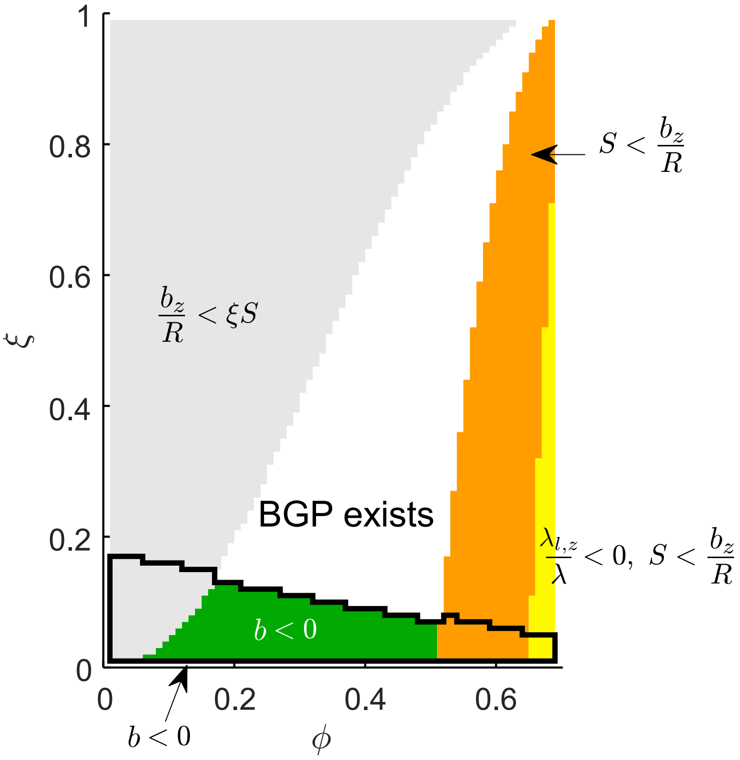

Figure 8: Domain of (ϕ, ξ) for BGP

the parameters (ϕ, ξ) as the white region, while the values of the other parameters are given as those for Japan in Table 2.

We exclude period 10 from the above minimization because our model aims to explain the long-term response rather than the immediate response to a debt shock. To calibrate the parameters, we choose the optimum parameters to minimize the distance between the simulation implied by our model and the actual data. This procedure is similar to impulse response matching, as described in, for example, Rotemberg and Woodford (1997), who chose the parameters to minimize the distance between the impulse responses implied by a reduced-form VAR and those implied by a DSGE model.[21]

As we saw in Figures 5–7, our model with the parameter values chosen by the above minimization exhibits an economic slowdown. However, economic growth may not slow if the values of the parameters are different. The white region in Figure 9 shows the values of $(\phi,\ \xi)$, with which economic growth is accelerated when the economy is hit by a debt shock with maximum z. An economic slowdown occurs only in the red region of Figure 9. In the white region, economic growth is accelerated because a larger amount of labor is used in the R&D sector than before. More labor is used for R&D activities because the marginal product of labor becomes lower in the production sector. When a debt shock

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[21] Our method can be regarded as a variant of impulse response matching if the actual data are interpreted as a response to a one-time shock of debt buildup.

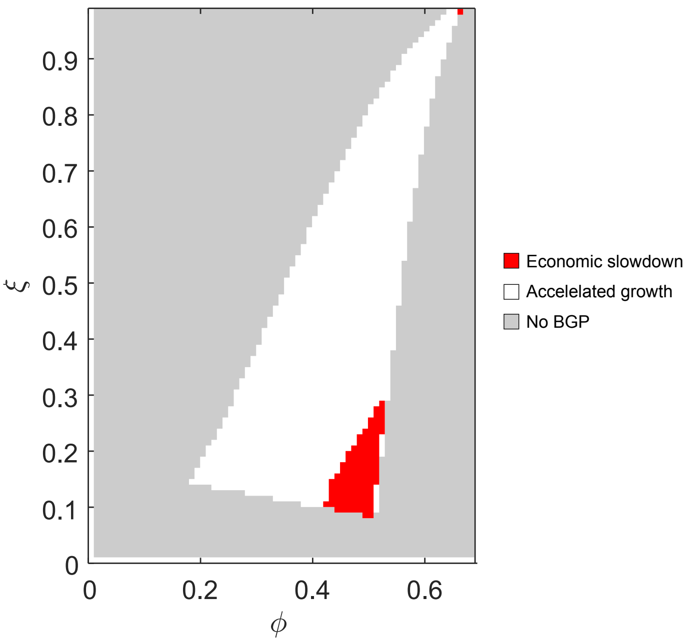

Figure 9: Domain of (ϕ, ξ) for economic slowdown

hits the economy, the marginal product of labor in the R&D sector can become higher or lower than that in the production sector, depending on the values of the parameters (ϕ, ξ).